Chapter 6, Section 5

Transformations of Variables

Method of Moment-Generating Functions

Method of Moment-Generating Functions

The crucial theorem is:

Theorem 6.1 (p. 318): If X and Y are random variables which both have moment-generating functions, and if

$$m_X(t) = m_Y(t) \text{ for all } t \text{ in some interval around } t = 0,$$

then X and Y have the same probability distribution.

Method of Moment-Generating Functions

Some other useful facts:

1. If $U = aY + b$, then

$$m_U(t) = E(e^{tU}) = E\left(e^{aYt+bt}\right) = e^{bt}E\left(e^{Y(at)}\right) = e^{bt}m_Y(at).$$

2. If $Y_1, Y_2, \ldots, Y_n$ are independent and $U = Y_1 + Y_2 + \cdots + Y_n$, then

$$m_U(t) = E(e^{tU}) = E\left(e^{tY_1+tY_2+\ldots+tY_n}\right) = E\left(e^{tY_1}e^{tY_2}\ldots e^{tY_n}\right)$$

$$= E\left(e^{tY_1}\right)E\left(e^{tY_2}\right)\ldots E\left(e^{tY_n}\right) \quad \text{by independence}$$

$$= m_{Y_1}(t)\, m_{Y_2}(t) \cdots m_{Y_n}(t)$$

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Method of Moment-Generating Functions

Some other useful facts:

3. If $Y_1, Y_2, \ldots, Y_n$ are independent and $U = \sum_{i=1}^{n} a_i Y_i$,

   then $m_U(t) = \prod_{i=1}^{n} m_{Y_i}(a_i\, t).$

4. If $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed (iid) with common distribution Y, i.e., a Random Sample from Y, and $U = Y_1 + Y_2 + \cdots + Y_n$, then $m_U(t) = \left[m_Y(t)\right]^n$.

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Method of Moment-Generating Functions

Example 1. Suppose that $Y_1, Y_2, \ldots, Y_m$ are independent binomial RVs with $Y_i \sim \text{bin}(n_i, p)$ [same p]. Then for $i = 1, 2, \ldots, m$,

$$m_{Y_i}(t) = \left(q + pe^t\right)^{n_i}$$

Let $Y = \sum_{i=1}^{m} Y_i$. Then by Property 2 on slide 3,

$$m_Y(t) = \prod_{i=1}^{m} m_{Y_i}(t) = \left(q + pe^t\right)^{n_1+n_2+\ldots+n_m},$$

so Y has a binomial distribution with $n = \sum_{i=1}^{m} n_i$ trials and probability of success, p. Thus, **the sum of independent binomials with the same probability of success, *p*, is also binomial.**

Example 2. Let $Y \sim \text{NegBin}(r, p)$,

$X_0 = 0$, and

X_i = # of trial on which i^{th} success occurs.

Then

$Y_i = X_i - X_{i-1} \sim \text{Geom}(p)$ for $i = 1, 2, \ldots, r$, and

$Y_1, Y_2, \ldots, Y_r$ are independent.

Also,

$$\sum_{i=1}^{r} Y_i = \underbrace{\sum_{i=1}^{r} (X_i - X_{i-1})}_{\text{telescoping sum}} = X_r - X_0 = X_r - 0 = Y.$$

Thus, $m_Y(t) = \prod_{i=1}^{r} m_{Y_i}(t) = \left(\dfrac{pe^t}{1 - qe^t}\right)^r$ by Property 2 on slide 3.

We have found the moment-generating functions of the negative binomial distributions using those of the geometric distributions.

Example 2. $Y \sim \text{NegBin}(r, p)$, $m_Y(t) = \left(\dfrac{pe^t}{1-qe^t} \right)^r$.

We can now use the mgf of Y to find its mean and variance.

For example,

$$E(Y) = m'_Y(0) = r\left(\frac{pe^0}{1-qe^0} \right)^{r-1} \cdot \frac{(pe^0)(1-qe^0)+(pe^0)(qe^0)}{(1-qe^0)^2}$$

$$= r\left(\frac{p}{1-q} \right)^{r-1} \cdot \frac{(p)(1-q)+pq}{(1-q)^2}$$

$$= r\left(\frac{p}{p} \right)^{r-1} \cdot \frac{p(p+q)}{p^2}$$

$$= \frac{r}{p}.$$

Example 3. Let $Y \sim \text{N}(\mu, \sigma^2)$ and $Z = \dfrac{Y-\mu}{\sigma} = \dfrac{1}{\sigma}Y - \dfrac{\mu}{\sigma}$.

Then $m_Y(t) = \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)$,

so $m_Z(t) = \exp\left(-\dfrac{\mu}{\sigma}t\right) m_Y\left(\dfrac{1}{\sigma}t\right)$ by Property 1 on slide 3

$$= \exp\left(-\frac{\mu}{\sigma}t\right)\exp\left(\mu\left(\frac{1}{\sigma}t\right) + \tfrac{1}{2}\sigma^2\left(\frac{1}{\sigma}t\right)^2\right)$$

$$= \exp\left(-\frac{\mu t}{\sigma} + \frac{\mu t}{\sigma} + \tfrac{1}{2}\left(\frac{\sigma^2 t^2}{\sigma^2}\right)\right) = \exp\left(\tfrac{1}{2}t^2\right)$$

So $Z \sim \text{N}(0, 1)$; Z has a standard normal distribution.

Transforming a random variable Y by subtracting its mean and dividing by its standard deviation is called **standardizing** Y.

Example 4. Let $Z \sim N(0,1)$ and $Y = \mu + \sigma Z$.

Then $m_Z(t) = \exp\left(\frac{1}{2} t^2\right)$,

so $m_Y(t) = \exp(\mu t) m_Z(\sigma t)$ by Property 1 on slide 3

$$= \exp(\mu t) \exp\left(\frac{1}{2} (\sigma t)^2\right)$$
$$= \exp\left(\mu t + \frac{1}{2} \sigma^2 t^2\right)$$

So $Y \sim N(\mu, \sigma^2)$; Y has a normal distribution with mean μ and variance σ^2.

Transforming a standard normal random variable Z in this fashion is a way of simulating other normal random variables.

Example 5. If $Y_i \sim N(\mu_i, \sigma_i^2)$ are independent for $i = 1, 2, \ldots, n$,

then $m_{Y_i}(t) = \exp\left(\mu_i t + \frac{1}{2} \sigma_i^2 t^2\right)$ for $1 \le i \le n$. Let $Y = \sum_{i=1}^{m} a_i Y_i$.

Then by the Property 3 on slide 4,

$$m_Y(t) = \prod_{i=1}^{m} m_{Y_i}(a_i t) = \prod_{i=1}^{m} \exp\left(\mu_i a_i t + \frac{1}{2} \sigma_i^2 (a_i t)^2\right)$$
$$= \exp\left(t \sum_{i=1}^{m} a_i \mu_i + \frac{1}{2} t^2 \sum_{i=1}^{m} a_i^2 \sigma_i^2 \right),$$

so $Y \sim N\left(\sum_{i=1}^{m} a_i \mu_i, \ \sum_{i=1}^{m} a_i^2 \sigma_i^2\right)$. Thus, **a linear combination of independent normal random variables is also normal.**

Special Case. Suppose $Y_1, Y_2, \ldots, Y_n$ are a random sample from a N(μ, σ^2) - distribution, and $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, the sample mean.

Then by Example 5, $\overline{Y} \sim N\left(\mu, \sigma^2/n\right)$.

Thus the sample mean of a random sample of size n from a normal distribution also has a normal distribution.

Its mean is the same as that of the original distribution and its variance is smaller – σ^2/n instead of σ^2.

This will be used often next semester.

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Example 6. (p. 319, Example 6.11)

Let $Z \sim$ N(0,1), so $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, and let $Y = Z^2$. Then

$$m_Y(t) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2t)z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/\left(2(1-2t)^{-1}\right)} dz$$

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Example 6. (p. 319, Example 6.11)

Let $Z \sim \mathrm{N}(0,1)$, so $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, and let $Y = Z^2$. Then

$$m_Y(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/\left(2(1-2t)^{-1}\right)} dz .$$

The integrand is proportional to the density function of the normal distribution with $\mu = 0$ and $\sigma^2 = (1 - 2t)^{-1}$. Thus, if we multiply by $\frac{1}{\sigma} = (1-2t)^{1/2}$ inside the integral and by $\sigma = (1-2t)^{-1/2}$ outside, we obtain $m_Y(t) = (1-2t)^{-1/2} \cdot 1 = (1-2t)^{-1/2}$.

Thus, $Y = Z^2 \sim \chi^2(1)$.

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