Definition. If Y_1 and Y_2 are jointly distributed discrete random variables with probability function $p(y_1, y_2)$, then the **marginal probability functions** of Y_1 and Y_2 are given by

$$p_1(y_1) = \sum_{y_2} p(y_1, y_2) \quad \text{and} \quad p_2(y_2) = \sum_{y_1} p(y_1, y_2)$$

These are (univariate) probability functions:

1. each is nonnegative since $p(y_1, y_2)$ is;
2. $\sum_{y_1} p_1(y_1) = \sum_{y_1}\sum_{y_2} p(y_1, y_2) = 1;\quad \sum_{y_2} p_2(y_2) = \sum_{y_2}\sum_{y_1} p(y_1, y_2) = 1.$

The distributions which they describe are called the **marginal distributions** of Y_1 and Y_2, respectively.

If $p(y_1, y_2)$ is given by a table, then $p_1(y_1)$ and $p_2(y_2)$ can be obtained by summing the columns and rows, and writing the sums in the “margins” of the table (hence the name).

2

Example. $(Y_1, Y_2) \sim p(y_1, y_2) = \begin{cases} \dfrac{y_1 + y_2}{21}, & y_1 = 1, 2, 3; \quad y_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}$

Then the marginal distributions of Y_1 and Y_2, respectively, are

$$p(y_1) = \begin{cases} \sum_{y_2} p(y_1, y_2) = \sum_{y_2=1}^{2} \dfrac{y_1 + y_2}{21} = \dfrac{y_1+1}{21} + \dfrac{y_1+2}{21} = \dfrac{2y_1+3}{21}, & y_1 = 1, 2, 3 \\ 0, & \text{elsewhere} \end{cases}$$

$$p_2(y_2) = \begin{cases} \sum_{y_1} p(y_1, y_2) = \sum_{y_1=1}^{3} \dfrac{y_1 + y_2}{21} = \dfrac{1+y_2}{21} + \dfrac{2+y_2}{21} + \dfrac{3+y_2}{21} = \dfrac{6+3y_2}{21}, & y_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}$$

3

Example. $(Y_1, Y_2) \sim p(y_1, y_2) = \begin{cases} \frac{y_1 + y_2}{21}, & y_1 = 1, 2, 3; \quad y_2 = 1, 2 \\ 0 \quad , & \text{elsewhere} \end{cases}$

Using a table, we can find the marginal distributions as follows:

y_2 \ y_1	1	2	3	$p_2(y_2)$
1	2 / 21	3 / 21	4 / 21	9 / 21
2	3 / 21	4 / 21	5 /21	12 / 21
$p_1(y_1)$	5 / 21	7 / 21	9 / 21	1

Exercise. Find the marginal distributions in the examples on slides 4 - 6 of Section 5.2.

4

Definition. If $(Y_1, Y_2) \sim p(y_1, y_2)$ as in the previous definition, then the **conditional discrete probability function** of Y_1 given $Y_2 = y_2$ is

$$\mathbf{p(y_1 \mid y_2)} = P(Y_1 = y_1 \mid Y_2 = y_2) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p(y_1, y_2)}{p_2(y_2)}$$

provided that $p_2(y_2) > 0$. $p(y_1 \mid y_2)$ is undefined if $p_2(y_2) = 0$.

We define $p(y_2 \mid y_1)$ in a similar fashion.

5

$p(y_1 \mid y_2)$ $= P(Y_1 = y_1 \mid Y_2 = y_2) = \dfrac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \dfrac{p(y_1, y_2)}{p_2(y_2)}$

$p(y_1 \mid y_2)$ is a (univariate) probability function:

1. $p(y_1 \mid y_2)$ is nonnegative since both $p(y_1, y_2)$ and $p_2(y_2)$ are; and

2. $\sum_{y_1} p(y_1 \mid y_2) = \sum_{y_1} \dfrac{p(y_1, y_2)}{p_2(y_2)} = \dfrac{1}{p_2(y_2)} \sum_{y_1} p(y_1, y_2) = \dfrac{1}{p_2(y_2)} p_2(y_2) = 1.$

The distribution $p(y_1 \mid y_2)$ describes is called the **conditional distribution of Y_1 given $Y_2 = y_2$** and is often denoted by $(Y_1 \mid y_2)$.

6

Example. $(Y_1, Y_2) \sim p(y_1, y_2) = \begin{cases} \dfrac{y_1 + y_2}{21}, & y_1 = 1, 2, 3; \quad y_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}$

We Know $p_1(y_1) = \begin{cases} \dfrac{2y_1 + 3}{21}, & y_1 = 1, 2, 3 \\ 0, & \text{elsewhere} \end{cases}$ and $p_2(y_2) = \begin{cases} \dfrac{6 + 3y_2}{21}, & y_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}$.

Using these formulas, we can get formulas for $p(y_1 \mid y_2)$ and $p(y_2 \mid y_1)$:

$$p(y_1 \mid y_2) = \begin{cases} \dfrac{p(y_1, y_2)}{p_2(y_2)} = \dfrac{(y_1 + y_2)/21}{(6 + 3y_2)/21} = \dfrac{y_1 + y_2}{6 + 3y_2}, & y_1 = 1, 2, 3 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{for } y_2 = 1, 2$$

$$p(y_2 \mid y_1) = \begin{cases} \dfrac{p(y_1, y_2)}{p_1(y_1)} = \dfrac{(y_1 + y_2)/21}{(2y_1 + 3)/21} = \dfrac{y_1 + y_2}{2y_1 + 3}, & y_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{for } y_1 = 1, 2, 3$$

Note: $p(y_1 \mid y_2)$ is only defined for $y_2 = 1, 2$, and $p(y_2 \mid y_1)$ only for $y_1 = 1, 2, 3$.

7

Example. $(Y_1, Y_2) \sim p(y_1, y_2) = \begin{cases} \dfrac{y_1 + y_2}{21}, & y_1 = 1, 2, 3; \quad y_2 = 1, 2 \\ 0 \quad , & \text{elsewhere} \end{cases}$

y_2 \ y_1	1	2	3	$p_2(y_2)$
1	2 / 21	3 / 21	4 / 21	9 / 21
2	3 / 21	4 / 21	5 /21	12 / 21
$p_1(y_1)$	5 / 21	7 / 21	9 / 21	1

We can obtain the probability function of $(Y_1 \mid y_2)$ using tables by dividing each entry in the y_2 row by the y_2 row sum; similarly for $(Y_2 \mid y_1)$ and the y_1 column.

y_1	$p(y_1 \mid 1)$
1	$\frac{2/21}{9/21} = \frac{2}{9}$
2	$\frac{3/21}{9/21} = \frac{3}{9}$
3	$\frac{4/21}{9/21} = \frac{4}{9}$
sum	1

y_1	$p(y_1 \mid 2)$
1	$\frac{3/21}{12/21} = \frac{3}{12}$
2	$\frac{4/21}{12/21} = \frac{4}{12}$
3	$\frac{5/21}{12/21} = \frac{5}{12}$
sum	1

y_2	$p(y_2 \mid 1)$
1	$\frac{2/21}{5/21} = \frac{2}{5}$
2	$\frac{3/21}{5/21} = \frac{3}{5}$
sum	1

y_2	$p(y_2 \mid 2)$
1	$\frac{3/21}{7/21} = \frac{3}{7}$
2	$\frac{4/21}{7/21} = \frac{4}{7}$
sum	1

y_2	$p(y_2 \mid 3)$
1	$\frac{4/21}{9/21} = \frac{4}{9}$
2	$\frac{5/21}{9/21} = \frac{5}{9}$
sum	1

8

Definition. Let Y_1 and Y_2 be continuous random variables with joint density function $f(y_1, y_2)$. The **marginal density functions** of Y_1 and Y_2 are defined by

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2\,, \quad -\infty < y_1 < \infty$$

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\,, \quad -\infty < y_2 < \infty$$

These are (univariate) density functions:

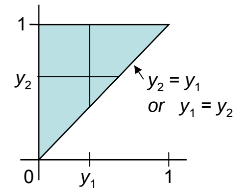
1. each is nonnegative since $f(y_1, y_2)$ is;
2. $\int_{-\infty}^{\infty} f_1(y_1) dy_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2) dy_2\, dy_1 = 1, \quad -\infty < y_1 < \infty;$ and
3. $\int_{-\infty}^{\infty} f_2(y_2) dy_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2) dy_1\, dy_2 = 1, \quad -\infty < y_2 < \infty.$

The distributions which they describe are called the **marginal distributions** of Y_1 and Y_2 respectively.

9

Example. (p. 244 #5.27a) Find the marginal distributions of Y_1 and Y_2 if

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$



Always **draw** the joint support of Y_1 and Y_2.

The support of Y_1 is $0 \le y_1 \le 1$, **and if $0 \le y_1 \le 1$**,

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_2 = \int_{y_1}^{1} (6 - 6y_2)\,dy_2 = \Big[6y_2 - 3y_2^2\Big]_{y_1}^{1}$$

$$= 3 - (6y_1 - 3y_1^2) = 3(1 - y_1)^2$$

Similarly, the support of Y_2 is $0 \le y_2 \le 1$, and if **$0 \le y_2 \le 1$**,

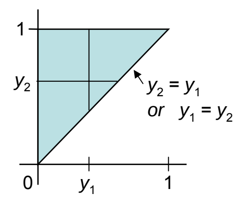
$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_1 = \int_0^{y_2} 6(1-y_2)\,dy_1 = \Big[6(1-y_2)y_1\Big]_{y_1=0}^{y_1=y_2}$$

$$= 6y_2(1 - y_2)$$

10

Example. (p. 244 #5.27a) Find the marginal distributions of Y_1 and Y_2 if

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$



So the marginal density functions of Y_1 and Y_2 are

$$f_1(y_1) = \begin{cases} 3(1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_2(y_2) = \begin{cases} 6y_2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

11

Now we would like to define conditional distributions in the continuous case. For any jointly distributed random variables, Y_1 and Y_2, we define the **conditional distribution function** of Y_1 given $Y_2 = y_2$ to be $\boldsymbol{F(y_1 \mid y_2)} = P(Y_1 \le y_1 \mid Y_2 = y_2)$.

For Y_1 and Y_2 discrete, $F(y_1 \mid y_2) = \sum_{t_1 \le y_1} p(t_1 \mid y_2)$

In the continuous case, we would like to get a density function, $f(y_1 \mid y_2)$, with the property that $F(y_1 \mid y_2) = \int_{-\infty}^{y_1} f(t_1 \mid y_2) dt_1$.

Since in the discrete case, the conditional probability function is $p(y_1 \mid y_2) = \dfrac{p(y_1, y_2)}{p_2(y_2)}$, it seems natural to try $f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$, provided $f_2(y_2) > 0$.

12

Before we can define $f(y_1 \mid y_2)$ in this manner, we need to check:

1. it is a density function (nonnegative and integrates to 1); and
2. It is the pdf of Y_1 given $Y_2 = y_2$; i.e., that

(∗) $F(y_1 \mid y_2) = P(Y_1 \le y_1 \mid Y_2 = y_2) = \int_{-\infty}^{y_1} f(t_1 \mid y_2) dt_1$.

But:

1. $f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$ is a density function since
   a. it is nonnegative as both $f(y_1, y_2)$ and $f_2(y_2)$ are; and
   b. $\int_{-\infty}^{\infty} f(y_1 \mid y_2) dy_1 = \int_{-\infty}^{\infty} \dfrac{f(y_1, y_2)}{f_2(y_2)} dy_1 = \dfrac{1}{f_2(y_2)} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$

$$= \frac{1}{f_2(y_2)} \cdot f_2(y_2) = 1.$$

2. See the handout on conditional distributions for a justification of (∗) and (2).

13

Definition. Let $(Y_1, Y_2) \sim f(y_1, y_2)$ be jointly continuous random variables with marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Then:

1. For any y_2 such that $f_2(y_2) > 0$, the conditional density of Y_1 given $Y_2 = y_2$ is $f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$; and

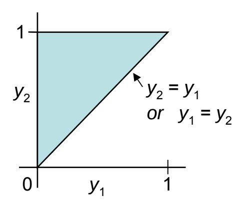
2. For any y_1 such that $f_1(y_1) > 0$, the conditional density of Y_2 given $Y_1 = y_1$ is $f(y_2 \mid y_1) = \dfrac{f(y_1, y_2)}{f_1(y_1)}$.

Note that $f(y_1 \mid y_2)$ is undefined if $f_2(y_2) = 0$ and $f(y_2 \mid y_1)$ is undefined if $f_1(y_1) = 0$.

14

Example. (p. 244 #5.27)

$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$



a. Find the marginal density functions for Y_1 and Y_2:

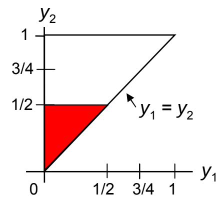
By our earlier work,

$$f_1(y_1) = \begin{cases} 3(1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_2(y_2) = \begin{cases} 6y_2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

15

Example. (p. 244 #5.27)



$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

b. Find $P\left(Y_2 \le \frac{1}{2} \,\middle|\, Y_1 \le \frac{3}{4}\right) = \dfrac{P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right)}{P\left(Y_1 \le \frac{3}{4}\right)}$.

[Cannot use conditional distributions, since given $Y_1 \le \frac{3}{4}$, not $Y_1 = \frac{3}{4}$.]

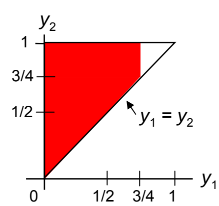
$$P\left(Y_2 \le \tfrac{1}{2}, Y_1 \le \tfrac{3}{4}\right) = \iint f(y_1, y_2)\,dA = \int_0^{1/2}\int_0^{y_2} 6(1-y_2)\,dy_1 dy_2$$

$$= \int_0^{1/2} \Big[6(1-y_2)y_1\Big]_{y_1=0}^{y_1=y_2} dy_2 = \int_0^{1/2} \Big[6(1-y_2)y_2 - 6(1-y_2)\cdot 0\Big] dy_2$$

$$= \int_0^{1/2} \left[6y_2 - 6y_2^2\right] dy_2 = 3y_2^2 - 2y_2^3 \Big|_0^{1/2} = \frac{1}{2}.$$

16

Example. (p. 244 #5.27)



$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

b. Find $P\left(Y_2 \le \frac{1}{2} \,\middle|\, Y_1 \le \frac{3}{4}\right) = \dfrac{P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right)}{P\left(Y_1 \le \frac{3}{4}\right)}$.

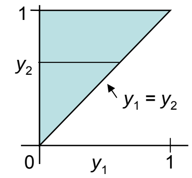
$$P\left(Y_1 \le \tfrac{3}{4}\right) = \int_{-\infty}^{3/4} f_1(y_1)\,dy_1 = \int_0^{3/4}\left(3 - 6y_1 + 3y_1^2\right)dy_1$$

$$= 3 - 6y_1 + 3y_1^2 \Big|_0^{3/4} = \frac{63}{64}.$$

Thus, $P\left(Y_2 \le \frac{1}{2} \,\middle|\, Y_1 \le \frac{3}{4}\right) = \dfrac{P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right)}{P\left(Y_1 \le \frac{3}{4}\right)} = \dfrac{1/2}{63/64} = \dfrac{32}{63}.$

17

Example. (p. 244 #5.27)



$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

c. Find the conditional density function of Y_1 given $Y_2 = y_2$:

$$f_2(y_2) = \begin{cases} 6y_2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$ so provided $f_2(y_2) > 0$ $(0 < y_2 < 1)$,

$$f(y_1 | y_2) = \begin{cases} \dfrac{f(y_1, y_2)}{f_2(y_2)} = \dfrac{6(1-y_2)}{6y_2(1-y_2)} = \dfrac{1}{y_2}, & 0 \le y_1 \le y_2 \\ 0, & \text{elsewhere} \end{cases}$$

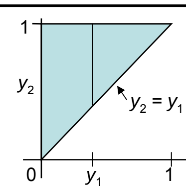
Notice that this is a uniform distribution on $[0, y_2]$; that is,

$(Y_1 \mid y_2) \sim \text{Uniform}(0, y_2)$ if $0 < y_2 < 1$.

$f(y_1 \mid y_2)$ is undefined for $y_2 \le 0$ or $y_2 \ge 1$.

18

Example. (p. 244 #5.27)



$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

d. Find the conditional density function of Y_2 given $Y_1 = y_1$:

$$f_1(y_1) = \begin{cases} 3(1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$ so provided $f_1(y_1) > 0$ $(0 \le y_1 < 1)$,

$$f(y_2 | y_1) = \begin{cases} \dfrac{f(y_1, y_2)}{f_1(y_1)} = \dfrac{6(1-y_2)}{3(1-y_1)^2} = \dfrac{2(1-y_2)}{(1-y_1)^2}, & y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

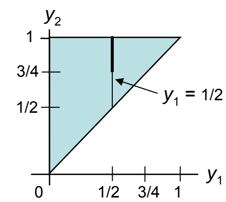
$f(y_2 \mid y_1)$ is undefined for $y_1 < 0$ or $y_1 \ge 1$.

19

Example. (p. 244 #5.27)

$$(Y_1, Y_2) \sim f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

$$f(y_2 \mid y_1) = \begin{cases} \dfrac{f(y_1, y_2)}{f_1(y_1)} = \dfrac{2(1 - y_2)}{(1 - y_1)^2}, & y_1 \le y_2 \le 1 \\ 0\,, & \text{elsewhere} \end{cases}$$



e. Find $P\left(Y_2 \ge \frac{3}{4} \mid Y_1 = \frac{1}{2}\right)$. Here we must use a conditional distribution:

$$P\left(Y_2 \ge \tfrac{3}{4} \mid Y_1 = \tfrac{1}{2}\right) = \int_{3/4}^{\infty} f\left(y_2 \mid \tfrac{1}{2}\right) dy_2$$

$$= \int_{3/4}^{1} \frac{2(1 - y_2)}{\left(1 - \frac{1}{2}\right)^2} dy_2 = \int_{3/4}^{1} \frac{2(1 - y_2)}{(1/4)} dy_2$$

$$= 8 \int_{3/4}^{1} (1 - y_2) dy_2 = \tfrac{1}{4}.$$

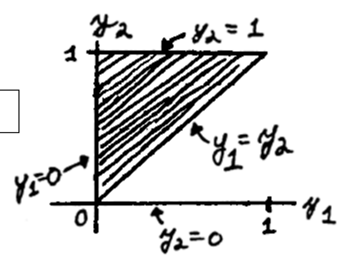
20

Example. Suppose Y_1 and Y_2 are jointly distributed with

$$f_2(y_2) = \begin{cases} c_2 y_2^4, & 0 < y_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

and for $0 < y_2 < 1$, $\quad$ $0 < y_1 < y_2 < 1$

$$f(y_1 \mid y_2) = \begin{cases} \dfrac{c_1 y_1}{y_2^2}, & 0 < y_1 < y_2 \\ 0\,, & \text{elsewhere.} \end{cases}$$



Draw the picture! Since $0 < y_2 < 1$ is the support of Y_2
and $0 < y_1 < y_2$ is the support of $(Y_1 \mid y_2)$,
the joint support of (Y_1, Y_2) is $0 < y_1 < y_2 < 1$, a triangle bounded
above by $y_2 = 1$,
to the left by $y_1 = 0$, and
below and to the right by $y_1 = y_2$.

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