

# Chapter 4, Section 9

## Continuous Random Variables

### Moment-Generating Functions

**Definitions.** Let  $Y$  be a continuous random variable with density function,  $f(y)$ , and let  $k = 1, 2, 3, \dots$ . Then:

$$\mu'_k = E[Y^k] = \int_{-\infty}^{\infty} y^k f(y) dy \text{ is the } k^{\text{th}} \text{ moment of } Y \text{ (about the origin).}$$

$$\mu_k = E[(Y - E(Y))^k] = E[(Y - \mu)^k] \text{ is the } k^{\text{th}} \text{ central moment of } Y.$$

**Notes.**  $\mu = E(Y) =$  the first moment of  $Y$  ( $= \mu'_1$ )

$$\mu_1 = E(Y - \mu) = 0$$

$$\begin{aligned} \sigma^2 &= V(Y) = \text{the second central moment of } Y \text{ } (= \mu_2) \\ &= E(Y^2) - [E(Y)]^2 = \mu'_2 - (\mu'_1)^2. \end{aligned}$$

**Definition.** If  $Y$  is a continuous random variable, its **moment-generating function (mgf)** is the function

$$m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f(y) dy$$

provided this function of  $t$  exists (converges) in some interval around 0 ("there exists a constant  $b > 0$  such that  $m(t)$  is finite for  $|t| \leq b$ ").

**Note.** When the mgf exists, it completely determines the distribution – two random variables with the same mgf have the same distribution.

Thus, if we find the mgf of a RV in a list of known mgfs, we have found the distribution of the RV.

We have also seen that the mgf of a discrete RV gives us an easy way to find its mean and variance; the same is true for continuous RVs.

**Why it is called the moment-generating function:**

As we have seen before for discrete random variables, and as the text proves on pp. 202-203 for continuous RVs,

$$m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f(y) dy = 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu'_k = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(Y^k)$$

and therefore we can derive (as in the discrete case) that

$$m^{(k)}(0) = \mu'_k = E(Y^k).$$

**Recall (#3.158):** If  $Y$  is a RV with MGF,  $m_Y(t)$ , and  $W = aY + b$  where  $a$  and  $b$  are constants, then  $m_W(t) = m_{aY+b}(t) = e^{bt} m_Y(at)$ . This is also true for mgfs of continuous random variables (p. 206 #4.137).

**Example.** If  $Y \sim N(\mu, \sigma^2)$ , find its mgf.

**Step 1.** Let  $Z \sim N(0, 1)$ . Then

$$\begin{aligned}
 m_Z(t) &= E(e^{tZ}) = \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z^2 - 2zt)}{2}\right) dz && \text{Complete the square} \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z^2 - 2zt + t^2)}{2} + \frac{t^2}{2}\right) dz && \text{Take out } e^{t^2/2} \\
 &= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-t)^2}{2}\right) dz && \text{The } N(t, 1)\text{-density} \\
 &= e^{t^2/2} \cdot 1
 \end{aligned}$$

so the mgf of the standard normal distribution is  $m_Z(t) = e^{t^2/2}$ .

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**Example.** If  $Y \sim N(\mu, \sigma^2)$ , find its mgf.

**Step 1.** Let  $Z \sim N(0, 1)$ . Then  $m_Z(t) = e^{t^2/2}$ .

**Step 2.** If  $Y \sim N(\mu, \sigma^2)$ , then  $Y = \sigma Z + \mu$ , so by problem 4.105,

$$\begin{aligned}
 m_Y(t) &= m_{\sigma Z + \mu}(t) = e^{\mu t} m_Z(\sigma t) \\
 &= e^{\mu t} e^{(\sigma t)^2/2} \\
 &= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)
 \end{aligned}$$

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**Example.** If  $Y \sim N(\mu, \sigma^2)$ , find  $E(Y)$  and  $V(Y)$ .

Since  $m(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$ , we obtain

$$m'(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot (\mu + \sigma^2 t)$$

$$m''(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot (\mu + \sigma^2 t)^2 + \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot (\sigma^2)$$

and therefore,

$$E(Y) = m'(0) = \exp\left(\mu \cdot 0 + \frac{\sigma^2 \cdot 0^2}{2}\right) \cdot (\mu + \sigma^2 \cdot 0) = 1 \cdot \mu = \mu$$

$$E(Y^2) = m''(0) = \exp(0) \cdot (\mu + \sigma^2 \cdot 0)^2 + \exp(0) \cdot (\sigma^2) = \mu^2 + \sigma^2$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

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