Chapter 3, Sections 1-2
Discrete Random Variables

Definition

Probability Distribution

Definitions

A **random variable** is a real-valued function, *Y*, whose domain is a sample space. Every time the experiment is performed, *Y* assumes one, and only one, *real numeric value*. So for each *s* in the sample space *S*, *Y*(*s*) is a real number.

The **support** of a random variable, *Y*, is the set of all real numbers that *Y* can (and does) actually assume: *Support*(*Y*) = { *Y*(*s*) | *s* ∈ *S* }

A **discrete random variable** is one whose support is either finite or countably infinite.

Theorem: The support of a *discrete* random variable *Y* is the set {*y* | the probability that *Y* takes on the value *y* is greater than zero}.

(*Note*: This theorem is **not true** for continuous random variables, which are discussed in chapter 4.)

Examples.

1. Toss 3 fair coins. *Y* = the number of heads. (Binomial)
   Support = { 0, 1, 2, 3 }. Sample Space has 8 elements:

s	*Y*(*s*)
HHH	*Y*(HHH) = 3
HHT	*Y*(HHT) = 2
HTH	*Y*(HTH) = 2
HTT	*Y*(HTT) = 1
THH	*Y*(THH) = 2
THT	*Y*(THT) = 1
TTH	*Y*(TTH) = 1
TTT	*Y*(TTT) = 0

Examples.

1. Toss 3 fair coins. *Y* = the number of heads. (Binomial)
   Support = { 0, 1, 2, 3 }
2. Toss a fair die until the first 6 appears (a "success").
   Y = # tosses it takes to get the first 6. (Geometric)
   Support = { 1, 2, 3, 4, . . . }
3. Toss a fair coin until the fourth head appears (a "success").
   Y = # tosses it takes to get the fourth head. (Negative Binomial)
   Support = { 4, 5, 6, 7, . . . }
4. Select 10 balls at random from a jar with 100 red and 80 blue balls, without replacement and without regard to order.
   Y = the number of red balls chosen. (Hypergeometric)
   Support = { 0, 1, 2, 3, . . . , 10 }
5. *Y* = the number of calls entering a telephone company switching office during a particular one minute period. (Poisson)
   Support = { 0, 1, 2, 3, . . . }
6. Toss a fair die. *Y* = the number showing on the top face.
   Support = { 1, 2, 3, 4, 5, 6 } (Discrete Uniform)

Notation

If a is a real number,
A is a subset of the real numbers, and
Y is a random variable,
then

$P(Y = a)$ is the probability of the **event** "Y assumes the value a."

This event is the set $\{s \in S \mid Y(s) = a\}$, so

$$P(Y = a) = P(\{s \in S \mid Y(s) = a\}).$$

$P(Y \in A)$ is the probability of the **event** "Y assumes a value in A."

This event is the set $\{s \in S \mid Y(s) \in A\}$, so

$$P(Y \in A) = P(\{s \in S \mid Y(s) \in A\}).$$

Definition. The **probability function** of a *discrete* random variable, Y, is the function $p(y) = P(Y = y)$.
Its domain is the set of all real numbers.

Calculations of the Probabilities $P(Y = y)$ and $P(Y \in A)$ in Discrete Sample Spaces

If the sample space, S, is ***discrete***, we can compute these probabilities by first determining the sample points in the corresponding event. For example, if the event $(Y = y)$ is $E = \{s_1, s_2, \ldots, s_m\}$ (we allow $m = \infty$)

$$p(y) = P(Y = y) = P(E) = \sum_{i=1}^{m} P(s_i) = \sum_{\substack{s \in S \\ Y(s) = y}} P(s)$$

Similarly, if A is a subset of the real numbers, then

$$P(Y \in A) = \sum_{y \in A} P(Y = y) = \sum_{\substack{s \in S \\ Y(s) \in A}} P(s)$$

Definition. The **probability function** of a *discrete* random variable, Y, is the function $p(y) = P(Y = y)$, with domain = set of **all** real numbers.

Example 1. Toss 3 fair coins. Y = # of heads. Support = {0, 1, 2, 3}. Find $p(y)$ for all y.

y	$p(y)$
0	
1	
2	
3	
sum	

$p(y) = 0$ if $y \neq 0, 1, 2, 3$.

s	$Y(s)$	$P(s)$
HHH	Y(HHH) = 3	1/8
HHT	Y(HHT) = 2	1/8
HTH	Y(HTH) = 2	1/8
HTT	Y(HTT) = 1	1/8
THH	Y(THH) = 2	1/8
THT	Y(THT) = 1	1/8
TTH	Y(TTH) = 1	1/8
TTT	Y(TTT) = 0	1/8

Definition. The **probability function** of a *discrete* random variable, Y, is the function $p(y) = P(Y = y)$, with domain = set of **all** real numbers.

Example 2. Toss a fair die until the first 6 appears (a "success").
Y = # tosses it takes to get the first 6. Support = {1, 2, 3, . . . }.
(a) Find a formula for $p(y)$. (b) Find $P(Y \leq 3)$.

Solution: (a) **Picture it** – think of filling blanks. If y = 1, 2, 3, . . .

toss	#1	#2	#3	#4	. . .	#(y–1)	#y
result	non-6	non-6	non-6	non-6	. . .	non-6	6
event	A	B	C	D	. . .		

Use the multiplication rule for probability:

$$P(A \cap B \cap C \cap D \cap \cdots) = P(A)\, P(B|A)\, P(C|A \cap B)\, P(D|A \cap B \cap C) \cdots$$

$$p(y) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdots \frac{5}{6} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^{y-1} \left(\frac{1}{6}\right)$$

$p(y) = 0$ if $y \neq 1, 2, 3, \ldots$

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Solution: (a) $p(y) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdots \frac{5}{6} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^{y-1}\left(\frac{1}{6}\right)$ for $y = 1, 2, 3, \ldots$.

(b) $P(Y \le 3) = p(1) + p(2) + p(3)$

$$= \left[\left(\tfrac{5}{6}\right)^0 + \left(\tfrac{5}{6}\right)^1 + \left(\tfrac{5}{6}\right)^2\right]\left(\tfrac{1}{6}\right)$$

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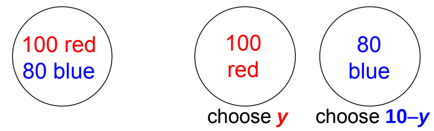
Definition. The **probability function** of a *discrete* random variable, Y, is the function $p(y) = P(Y = y)$, with domain = set of **all** real numbers.

Example 4. Choose 10 balls from a jar with 100 red and 80 blue balls, without replacement and without regard to order.
Y = the number of red balls chosen. Support = { 0, 1, 2, . . . , 10 }.
Find $p(y)$ for all y.

Solution. Let E be the event $Y = y$. Then $p(y) = P(Y = y) = P(E) = \frac{|E|}{|S|}$
where the sample space S is the set of all selections of 10 balls from the jar.



Calculate $|S| = \binom{180}{10}$, $|E| = \binom{100}{y}\binom{80}{10-y}$, so $p(y) = \dfrac{\binom{100}{y}\binom{80}{10-y}}{\binom{180}{10}}$
for y = 0, 1, 2, . . . , 10, and $p(y) = 0$ elsewhere.

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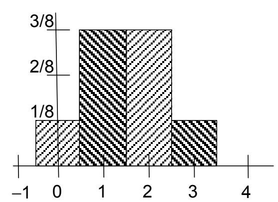
Example. Table the probability function, $p(y)$, for the random variable Y = # heads on 3 tosses of a fair coin in Example 1. Then draw its probability histogram.

From our previous work, we obtain the table at the right:

y	0	1	2	3	sum
$p(y)$	1/8	3/8	3/8	1/8	1

Note: This tables $p(y)$ <u>only</u> for $y \in$ Support(Y) - those y for which $p(y) \neq 0$; however $p(y)$ is defined for ***all*** real numbers, y.

The ***probability histogram***, a graphical description of $p(y)$, is at the right.
In a probability histogram of a discrete random variable, Y, the area in each bar is the probability that Y takes on the value over which the bar is centered.



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Theorem. If Y has a discrete probability distribution with probability function, $p(y)$, then

(1) $0 \le p(y) \le 1$, for all y; and

(2) $\sum_y p(y) = 1$, where the sum is over all values of y with nonzero probability (the support of Y).

Note. $p(y)$ is also called the **probability mass function** or the **discrete density function** of Y.

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Example. (p. 91 #3.10) A particular item is rented, on the average, only one day in five. If rental on one day is independent of rental on any other day, find the probability distribution of Y, the number of days between a pair of successive rentals (between one rental and the next rental).

Solution. Let R denote the event that a rental occurs on a given day, and N denote the event that a rental does not occur on a given day.

The RV, Y, is related to a sequence $RNNN\ldots NR$: Y = # of Ns.

To get the probability function of Y, we need to calculate the probabilities of all possible sequences of the form $NNNNN\ldots NR$, where the number of Ns ranges from 0 on up; that is, the support of Y is { 0, 1, 2, 3, . . . }.

We ignore the initial R in the original sequence since we are considering the number of days between rentals. Therefore, there was a rental on the first day; however for each succeeding day, including the last day in the sequence, we have to take into account the two possibilities of rental or non-rental.

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Solution. Let R denote the event that a rental occurs on a given day, and N denote the event that a rental does not occur on a given day.

The RV, Y, is related to a sequence $RNNN\ldots NR$: Y = # of Ns.

The support of Y is { 0, 1, 2, 3, . . . } and the probability function of Y is

$$P(Y = y) = \begin{cases} (0.8)^y\,(0.2), & y = 0, 1, 2, 3, \ldots \\ 0, & \text{elsewhere}. \end{cases}$$

The random variable Y is similar to the geometric random variable in Example 2, except that it counts the number of failures (Ns) **before** the first success, rather than the number of trials up to and including the first success (R). Some take this as the definition of a geometric random variable (more about this later).