Slide 11: p. 61, #2.102. In a certain population, 10% will contract disease I at some time in their lifetime, 15% will contract disease II, and 3% will contract both.

Let A denote the event that a randomly chosen person contracts disease I and let B denote the event that a randomly chosen person contracts disease II.

Then we are told $P(A) = 0.10$

$P(B) = 0.15$

$P(A \cap B) = 0.03$.

a. Find the probability that a randomly chosen person will contract at least one of the diseases.

This asks for

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.10 + 0.15 - 0.03 = 0.22.$$

b. Given that a person has contracted at least one of the diseases, find the conditional probability that s/he will contract both.

This asks for

$$P(A \cap B \,|\, A \cup B) = \frac{P\big((A \cap B) \cap (A \cup B)\big)}{P(A \cup B)}.$$

Since $A \cap B \subseteq A \cup B$, it follows that $(A \cap B) \cap (A \cup B) = A \cap B$, so we obtain

$$P(A \cap B \,|\, A \cup B) = \frac{P\big((A \cap B) \cap (A \cup B)\big)}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0.03}{0.22} = \frac{3}{22} = 0.136.$$

Slide 14: p. 60, #2.92. We are told a lie detector concludes that a truthful person is lying with a probability 0.05, and that lie detector tests are mutually independent. Three people are tested.

a. What is the probability that the detector will conclude that all three are lying when all are telling the truth?

Let A_1, A_2, A_3 be the event that the detector concludes person 1, 2, 3, respectively, is lying. We are told all three are truthful, so $P(A_i) = 0.05$ for $i = 1, 2, 3$. We are asked for

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2)\, P(A_3) \quad \text{by mutual independence} \\ &= (0.05)\,(0.05)\,(0.05) \\ &= 0.000125. \end{aligned}$$

b. What is the probability that the detector will conclude that at least one person is lying when all are telling the truth?

We are asked for $P(A_1 \cup A_2 \cup A_3)$ where the A_i are as above and again all three are truthful. We could use Inclusion-Exclusion, but there is an easier way.

Because the A_i are mutually independent, so are their complements (and, in fact, all triples of events B_1, B_2, B_3 where each $B_i = A_i$ or $\overline{A_i}$; for example, A_1, $\overline{A_2}$ and A_3 are mutually independent), and with mutually independent events, it is easy to calculate the probability of their intersection. So use complements, DeMorgan's law, and mutual independence to get

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= 1 - P(\overline{A_1 \cup A_2 \cup A_3}) \quad \text{by the property of complements} \\ &= 1 - P(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}) \quad \text{by DeMorgan} \\ &= 1 - P(\overline{A_1})\,P(\overline{A_2})\,P(\overline{A_3}) \quad \text{by mutual independence.} \end{aligned}$$

But

$$P(\overline{A_i}) = 1 - P(A_i) = 1 - 0.05 = 0.95 \text{ for } i = 1, 2, 3,$$

so

$$P(A_1 \cup A_2 \cup A_3) = 1 - (0.95)^3 = 0.142625.$$

Slide 17. We are told that a string of 12 Christmas tree light bulbs has three defective bulbs and nine nondefective bulbs. They are tested, one at a time, until the third defective bulb is found.

a. Find the probability that the third defective bulb is found on the third test.

Let A_i denote the event that the i^{th} bulb tested is defective for i = 1, 2, 3, so that $\overline{A_i}$ is the event that the i^{th} bulb tested is nondefective. Then we want $P(A_1 \cap A_2 \cap A_3)$.

The A_i are **not** independent, so we **cannot say**

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)\,,$$

but we can use the general multiplication rule:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2).$$

Now $P(A_1) = \frac{3}{12}$ since three of the 12 bulbs are defective. On the second test, there will be 11 bulbs from which to choose and given that A_1 occurred, two of them will be defective, so

$$P(A_2 \mid A_1) = \tfrac{2}{11}.$$

On the third test, there will be 10 bulbs from which to choose, and given that $A_1 \cap A_2$ occurred, only one of them will be defective, so

$$P(A_3 \mid A_1 \cap A_2) = \tfrac{1}{10}.$$

Thus

$$P(A_1 \cap A_2 \cap A_3) = \tfrac{3}{12} \cdot \tfrac{2}{11} \cdot \tfrac{1}{10} = \tfrac{1}{220}.$$

We can also show this by thinking of filling three blanks with bulbs from the set $\{ D_1, D_2, D_3, N_1, N_2, \dots, N_9\}$ of bulbs. The sample space is the set of all possible ordered triples of bulbs, so

$$|S| = P(12, 3) = 12 \cdot 11 \cdot 10.$$

The event, E, of interest is the set of ordered triples of defective bulbs, so

$$|E| = P(3, 3) = 3! = 6,$$

and

$$P(E) = \frac{|E|}{|S|} = \frac{6}{12 \cdot 11 \cdot 10} = \frac{1}{120}.$$

b. Find the probability that the third defective bulb is found on the sixth test.

Think of the bulbs as being labeled $\{D_1, D_2, D_3, N_1, N_2, \ldots, N_9\}$ as above. Let F be the event of interest. Here the sample space, S, is the set of all ordered 6-tuples of bulbs, so

$$|S| = P(12, 6) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7.$$

The event, F, is the set of all ordered 6-tuples of bulbs containing three Ds and three Ns with a D as the sixth coordinate:

$$\underbrace{\_\ \_\ \_\ \_\ \_}_{\text{2 Ds \& 3 Ns}}\ \underline{D}$$

So $|F|$ is the number of ways to distribute two of the Ds and three of the Ns among the first five blanks and place the remaining D in the sixth blank. The task of placing two Ds and three Ns in the first five blanks can be done in three steps:

Step 1: choose two blanks for the Ds $\binom{5}{2} = 10$ ways to do it

Step 2: put Ds in these two blanks $P(3, 2) = 3 \cdot 2$ ways

Step 3: put Ns in the other three blanks $P(9, 3) = 9 \cdot 8 \cdot 7$ ways

The final step is placing a D in the sixth blank, and this can be done in only one way since there were a total of three defective bulbs and two have already been used. So by the multiplication rule,

$$|F| = \binom{5}{2} \cdot P(3,2) \cdot P(9,3) \cdot 1 = 10 \cdot 3 \cdot 2 \cdot 9 \cdot 8 \cdot 7 \cdot 1$$

and

$$P(F) = \frac{|F|}{|S|} = \frac{10 \cdot 3 \cdot 2 \cdot 9 \cdot 8 \cdot 7 \cdot 1}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} = \frac{1}{22} = 0.045.$$

Alternate solution to b. We have 12 bulbs, three defective and nine nondefective, and we test until the third defective bulb is found. We want the probability that it is found on the sixth test. This will be the same as the probability that if we test **<u>all</u>** 12 bulbs, one at a time, we find the third defective bulb on the sixth test. But this is the probability that if we test all 12 bulbs, the 6^{th} is defective (D) and the 7^{th}, 8^{th}, 9^{th}, 10^{th}, 11^{th}, and 12^{th} are all nondefective (N). By the generalized multiplication rule, this probability is:

$$P(12^{th} \text{ is } N) \cdot P(11^{th} \text{ is } N \mid 12^{th} \text{ is } N) \cdot P(10^{th} \text{ is } N \mid 11^{th}, 12^{th} \text{ are } N) \cdot P(9^{th} \text{ is } N \mid 10^{th}, 11^{th}, 12^{th} \text{ are } N)$$
$$\cdot P(8^{th} \text{ is } N \mid 9^{th}, 10^{th}, 11^{th}, 12^{th} \text{ are } N) \cdot P(7^{th} \text{ is } N \mid 8^{th}, 9^{th}, 10^{th}, 11^{th}, 12^{th} \text{ are } N)$$
$$\cdot P(6^{th} \text{ is } D \mid 7^{th}, 8^{th}, 9^{th}, 10^{th}, 11^{th}, 12^{th} \text{ are } N)$$
$$= \frac{9}{12} \cdot \frac{8}{11} \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{1}{22} = 0.045.$$

The fractions above come from thinking of creating all lists of 9 Ns and 3 Ds in reverse order, from 12^{th} place to 1^{st} place. Then the first six fractions are the probabilities a list has an N in position k given that it has Ns in all later positions for $k = 12, 11, 10, 9, 8, 7$; and the last fraction (3/6) is the probability a list has a D in position 6 given that it has Ns in all of positions 7 – 12.