NAME SOLUTIONS

ACSC/MATH 347/447 Final Exam December 17, 2007

Directions.

- **Show all your work.** *Full credit will* **not** *be given for any problem unless sufficient work is shown or a suitable explanation is given.*
- You may **not** use the cdf or pdf functions on a calculator in any problem.
- Part of the credit for each problem will be for style, readability, and mathematical correctness.
- The maximum score is 100 points.
- 2½ hours is allowed for the test.

1. (5 points) An urn contains twelve red and eight white balls. You draw six balls from the urn without replacement. Let the random variable Y denote the number of red balls drawn. Find the probability that $Y \geq 5$.

Y has a hypergeometric distribution with $N=20$, $r=12$ and $n=6$ using the text's notation.

$$P(Y \geq 5) = P(Y=5) + P(Y=6)$$

$$= \frac{\binom{12}{5}\binom{8}{1}}{\binom{20}{6}} + \frac{\binom{12}{6}\binom{8}{0}}{\binom{20}{6}} = \frac{(792)(8)}{38760} + \frac{(924)(1)}{38760}$$

$$= 0.1873$$

2. (10 points) You play backgammon regularly with the same opponent. You are a better player and ***win*** 75% of the time and ***lose*** 25% of the time. (You can assume that the games are independent.)

a. If you play 10 games tomorrow, what is the probability that you ***lose*** exactly 2 games?

Let Y = # games you lose tomorrow. Then $Y \sim \text{bin}(10, 0.25)$, so

$$P(Y=2) = \binom{10}{2}(0.25)^2(0.75)^8 = 0.2816$$

b. What is the probability that you play at least 7 games before your first *loss*?

Let X = # games up to and including your first loss. Then $X \sim \text{Geom}(0.25)$.

$$P(X > 7) = (0.75)^7 = 0.1335.$$

alternatively, this is the probability you win your first 7 games $= (0.75)^7$.

3. (12 points) Let Y be a continuous random variable with density function $f(y)=\begin{cases}3y^2, & 0\le y\le 1\\ 0, & \text{elsewhere.}\end{cases}$

a. Find $P\left(Y>\frac{1}{2}\right)$.

b. Find the mean and the variance of Y.

(a) $P(Y>\frac{1}{2}) = \int_{1/2}^{\infty} f(y)\,dy = \int_{1/2}^{1} 3y^2\,dy$

$= y^3\Big|_{1/2}^{1} = 1^3-(\frac{1}{2})^3 = \underline{\underline{\frac{7}{8}}}$

(b) $E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 (y)(3y^2)\,dy$

$= \frac{3y^4}{4}\Big|_0^1 = \frac{3}{4} - 0 = \underline{\underline{\frac{3}{4}}}$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 (y^2)(3y^2)\,dy$

$= \frac{3y^5}{5}\Big|_0^1 = \frac{3}{5} - 0 = \frac{3}{5}$

so $V(Y) = E(Y^2) - [E(Y)]^2$

$= \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{48-45}{80} = \underline{\underline{\frac{3}{80}}}$

4. (10 points) The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y, is uniformly distributed on the interval 1 to 5 days. The cost incurred following a board failure is related to the delivery time for a new board by the formula $C = 50 + 3Y + 9Y^2$.

   a. Find the probability that the delivery time, Y, exceeds 2.5 days.

   b. Find $E(C)$, the expected cost associated with the failure of a circuit board.

(a) $Y \sim U(1, 5)$, so

$$P(Y > 2.5) = \frac{5 - 2.5}{5 - 1} = \frac{2.5}{4} = \underline{\underline{0.625}}$$

(b) $E(C) = E(50 + 3Y + 9Y^2)$

$$= 50 + 3E(Y) + 9E(Y^2),$$

Since $Y \sim U(1, 5)$,

$$E(Y) = \frac{5+1}{2} = 3$$

$$V(Y) = \frac{(5-1)^2}{12} = \frac{16}{12} = \frac{4}{3}$$

So $E(Y^2) = V(Y) + [E(Y)]^2 = \frac{4}{3} + 3^2 = \frac{31}{3}$

and

$$E(C) = 50 + 3(3) + 9\left(\frac{31}{3}\right) = \underline{\underline{152}}.$$

5. (10 points) A discrete random variable, Y, takes values in the set $\{0, 1, 4, 9, 16\}$ and has probability function given by the table below.
   a. Find k.
   b. Find $E(\sqrt{Y})$.

y	0	1	4	9	16
$p(y)$	0.10	0.35	0.20	0.25	k

(a) $\sum_y p(y) = 1$, so

$0.10 + 0.35 + 0.20 + 0.25 + k = 1$

$0.90 + k = 1$

$k = \underline{\underline{0.10}}$

(b) $E(\sqrt{Y}) = \sum_y \sqrt{y}\, p(y)$

$= \sqrt{0}(0.10) + \sqrt{1}(0.35) + \sqrt{4}(0.20) + \sqrt{9}(0.25) + \sqrt{16}(0.10)$

$= 0 + 0.35 + 0.40 + 0.75 + 0.40$

$= 1.90$

6. (10 points) Bean seeds from supplier A have an 85% germination rate and those from supplier B have a 75% germination rate. A seed packaging company purchases 40% of its bean seeds from supplier A and 60% from supplier B and mixes these seeds together.

   a. Let G be the event that a seed selected at random from the mixed seeds will germinate. Find $P(G)$.

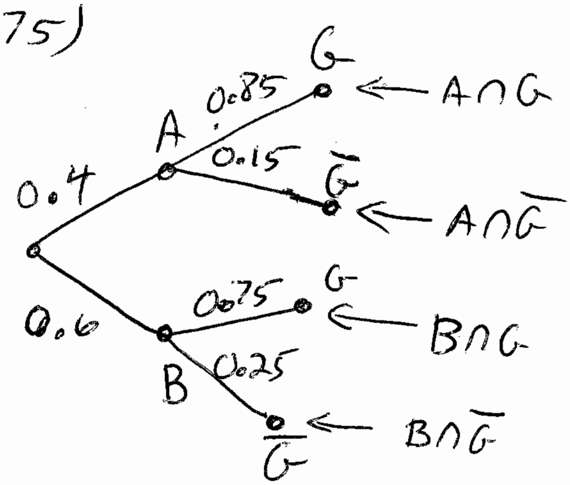
   b. Given that a seed germinates, find the conditional probability that the seed was purchased from supplier A.

(a) Given $P(A) = 0.40$, $P(B) = 0.60$

$P(G|A) = 0.85$, $P(G|B) = 0.75$

so by the Law of Total Probability,

$$P(G) = P(A)\,P(G|A) + P(B)\,P(G|B)$$
$$= (0.40)(0.85) + (0.60)(0.75)$$
$$= 0.34 + 0.45$$
$$= 0.79$$



(b) $$P(A|G) = \frac{P(A \cap G)}{P(G)}$$

$$= \frac{P(A)\,P(G|A)}{P(G)}$$

$$= \frac{(0.40)(0.85)}{0.79}$$

$$= \frac{0.34}{0.79} = 0.4304$$

7. (10 points) The random variable Y has a normal distribution with mean 3 and variance 4. In each part, *show your work or explain how you get your answer.* Use the normal tables, not a calculator.

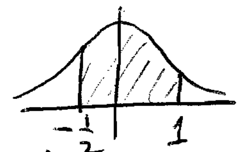
   a. Find $P(2 < Y < 5)$.

   b. Find the value c such that $P(Y \le c) = 0.9960$.

$Y \sim N(3,4)$, so $Z = \frac{Y-3}{2} \sim N(0,1)$.

(a) $P(2<Y<5) = P\left(\frac{2-3}{2} < \frac{Y-3}{2} < \frac{5-3}{2}\right)$

$= P(-0.5 < Z < 1)$



$= 1 - P(Z>1) - P(Z<-0.5)$

$= 1 - P(Z>1) - P(Z>0.5)$

$= 1 - 0.1587 - 0.3085$

$= \underline{0.5328}$

[Using SOA tables,

$P(2<Y<5) = P(-0.5<Z<1)$

$= P(Z<1) - P(Z<-0.5)$

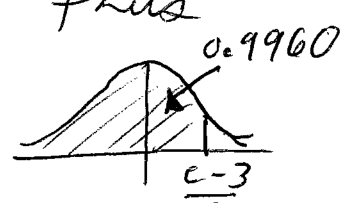
$= P(Z<1) - P(Z>0.5)$

$= P(Z<1) - (1 - P(Z \le 0.5))$

$= 0.8413 - 1 + 0.6915 = \underline{0.5328}$]

(b) $0.9960 = P(Y \le c) = P\left(Z \le \frac{c-3}{2}\right)$, so

$0.0040 = P\left(Z > \frac{c-3}{2}\right)$, and thus

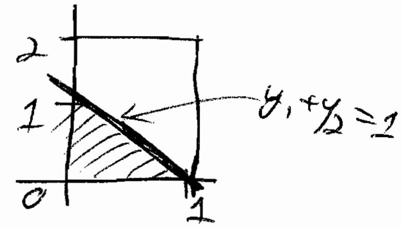


$\frac{c-3}{2} = 2.65$, so

$c = 3 + 2(2.65) = \underline{8.3}$

8. (15 points) Let Y_1 and Y_2 be continuous random variables with joint density function

$$f(y_1, y_2) = \begin{cases} \dfrac{2y_1 + y_2}{4}, & 0 < y_1 < 1, \quad 0 < y_2 < 2 \\ 0, & \text{elsewhere.} \end{cases}$$



a. Find $P(Y_1 + Y_2 \le 1)$. [Draw the picture!]

b. Find the marginal probability density function, $f_1(y_1)$, of Y_1.

c. Find the conditional probability density function, $f\left(y_2 \,\middle|\, \frac{1}{4}\right)$, of Y_2, given $Y_1 = \frac{1}{4}$.

(a) $P(Y_1 + Y_2 \le 1) = \iint_{\Delta} f(y_1, y_2)\, dA$

$= \int_0^1 \int_0^{1-y_2} \left(\frac{2y_1 + y_2}{4}\right) dy_1\, dy_2$

$= \frac{1}{4} \int_0^1 \left[y_1^2 + y_1 y_2\right]_0^{1-y_2} dy_2$

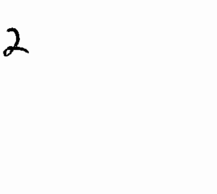
$= \frac{1}{4} \int_0^1 \left[(1-y_2)^2 + (1-y_2)y_2\right] dy_2$

$= \frac{1}{4} \int_0^1 \left(1 - 2y_2 + \cancel{y_2^2} + y_2 - \cancel{y_2^2}\right) dy_2$

$= \frac{1}{4} \int_0^1 (1 - y_2)\, dy_2 = \frac{1}{4}\left[y_2 - \frac{y_2^2}{2}\right]_0^1$

$= \frac{1}{4}\left[\left(1 - \frac{1}{2}\right) - (0 - 0)\right] = \underline{\frac{1}{8}}$

(b) $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$, so



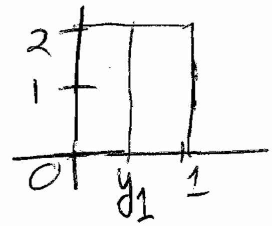
for $0 < y_1 < 1$, $f_1(y_1) = \int_0^2 \frac{2y_1 + y_2}{4}\, dy_2$

$= \frac{1}{4}\left(2y_1 y_2 + \frac{y_2^2}{2}\right)\Big|_{y_2=0}^{y_2=2}$

$= \frac{1}{4}(4y_1 + 2) = y_1 + \frac{1}{2}$

8. (15 points) Let Y_1 and Y_2 be continuous random variables with joint density function

$$f(y_1, y_2) = \begin{cases} \dfrac{2y_1 + y_2}{4}, & 0 < y_1 < 1, \quad 0 < y_2 < 2 \\ 0, & \text{elsewhere.} \end{cases}$$



a. Find $P(Y_1 + Y_2 \le 1)$. [Draw the picture!]

b. Find the marginal probability density function, $f_1(y_1)$, of Y_1.

c. Find the conditional probability density function, $f\left(y_2 \,\middle|\, \frac{1}{4}\right)$, of Y_2, given $Y_1 = \frac{1}{4}$.

(b) (continued)

Thus

$$f_1(y_1) = \begin{cases} y_1 + \frac{1}{2}, & 0 < y_1 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(c) $f(y_2 | \frac{1}{4}) = \dfrac{f(\frac{1}{4}, y_2)}{f_1(\frac{1}{4})}$, so

$$f(y_2 | \tfrac{1}{4}) = \begin{cases} \dfrac{\left(\frac{2(\frac{1}{4}) + y_2}{4}\right)}{\left(\frac{1}{4} + \frac{1}{2}\right)} = \dfrac{\frac{1}{2} + y_2}{3}, & 0 < y_2 < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$f(y_2 | \tfrac{1}{4}) = \begin{cases} \dfrac{2y_2 + 1}{6}, & 0 < y_2 < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

9. (10 points) Let Y be a random variable with density function $f(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$
Find the density function of $U = 2 - 2Y$.

Since the support of Y is $0 < y < 1$,

$$0 > -2y > -2$$
$$2 > u = 2 - 2y > 0$$

so the support of U is $0 < u < 2$.

Method of Distribution Functions: If $0 < u < 2$,

$$F_U(u) = P(U \le u) = P(2 - 2Y \le u) = P(-2Y \le u - 2)$$
$$= P(Y > 1 - \tfrac{u}{2}) = \int_{1-\frac{u}{2}}^{1} 2y\,dy$$
$$= y^2 \Big|_{1-\frac{u}{2}}^{1} = 1^2 - (1 - \tfrac{u}{2})^2, \text{ so}$$

$$F_U(u) = \begin{cases} 0, & u \le 0 \\ 1 - (1 - \frac{u}{2})^2, & 0 < u < 2 \\ 1, & u \ge 2 \end{cases}$$

and $f_U(u) = \begin{cases} -2(1 - \frac{u}{2})(-\frac{1}{2}) = 1 - \frac{u}{2}, & 0 < u < 2 \\ 0, & \text{elsewhere} \end{cases}$

Method of Transformations: Since $U = 2 - 2Y$ is a decreasing, differentiable function, we can apply this method. If $u = h(y) = 2 - 2y$, then $2y = 2 - u \Rightarrow y = 1 - \frac{u}{2} \Rightarrow \frac{dy}{du} = -\frac{1}{2}$.

Thus, $f_U(u) = f_Y(y) \cdot \left|\frac{dy}{du}\right| = \begin{cases} 2(1 - \frac{u}{2}) \cdot \left|-\frac{1}{2}\right| = 1 - \frac{u}{2}, & 0 < u < 2 \\ 0, & \text{elsewhere.} \end{cases}$

10. (8 points) Print requests arrive randomly and independently at a laser printer on a local area computer network. Let Y denote the time in seconds that it takes to process a request, and assume that Y is exponentially distributed with mean 10 seconds.

a. What is the probability that the next request will take more than 30 seconds to process?

$Y \sim Exp(\beta = 10)$, so $F(y) = 1 - e^{-y/10}, \ y > 0$

$S(y) = e^{-y/10}, \ y > 0$

$P(Y > 30) = 1 - F(30) = S(30) = e^{-30/10} = \underline{\underline{e^{-3}}}$

$= \underline{0.0498}$

b. Processing started on the request currently being printed 15 seconds ago. What is the probability that it will be finished within the next 10 seconds?

We are asked for $P(Y \leq 25 \mid Y \geq 15)$. Since the exponential random variables are memoryless, this is

$P(Y \leq 10) = F(10) = 1 - e^{-10/10} = \underline{\underline{1 - e^{-1}}}$

$= \underline{\underline{0.6321}}$

11. (8 points Extra credit) Suppose that Y_1 and Y_2 are random variables with

$$E(Y_1) = 2, \quad E(Y_2) = -7, \quad V(Y_1) = 2, \quad V(Y_2) = 4, \text{ and } Cov(Y_1, Y_2) = 2.$$

Find the mean and the variance of $U = 6Y_1 - 5Y_2$.

$E(U) = E(6Y_1 - 5Y_2) = 6E(Y_1) - 5E(Y_2)$

$= (6)(2) - (5)(-7) = \underline{\underline{47}}$

$V(U) = V(6Y_1 - 5Y_2)$

$= 6^2 V(Y_1) + (-5)^2 V(Y_2) + 2(6)(-5) Cov(Y_1, Y_2)$

$= (36)(2) + (25)(4) - 60(2)$

$= 72 + 100 - 120 = \underline{\underline{52}}$