

*Show all your work.***100 points maximum score****60 minutes time limit***The test is printed on both sides of the paper.*

1. (22=6+10+6 points) If  $Y$  has a normal distribution with mean 3 and variance 4, find:
- $P(Y > 6)$ .
  - $P(2.5 < Y < 4)$ .
  - The value  $c$  such that  $P(Y \leq c) = 0.9960$ .

$$Y \sim N(3, 4) \Rightarrow Z = \frac{Y - \mu}{\sigma} = \frac{Y - 3}{2} \sim N(0, 1)$$

$$(a) \ P(Y > 6) = P\left(Z > \frac{6-3}{2}\right) = P(Z > 1.5) = 0.0668$$

$$(b) \ P(2.5 < Y < 4) = P\left(\frac{2.5-3}{2} < Z < \frac{4-3}{2}\right)$$

$$= P(-0.25 < Z < 0.5)$$

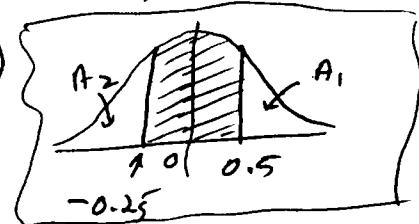
$$= 1 - A_1 - A_2$$

$$= 1 - P(Z > 0.5) - P(Z < -0.25)$$

$$= 1 - P(Z > 0.5) - P(Z > 0.25) \text{ by symmetry}$$

$$= 1 - 0.3085 - 0.4013$$

$$= 0.2902$$



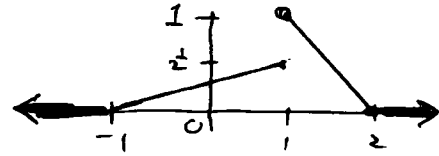
$$(c) \ 0.9960 = P(Y \leq c) = P\left(\frac{Y-3}{2} \leq \frac{c-3}{2}\right) = P\left(Z \leq \frac{c-3}{2}\right)$$

$$\Rightarrow P\left(Z > \frac{c-3}{2}\right) = 1 - 0.9960 = 0.0040$$

$$\Rightarrow \frac{c-3}{2} = 2.65 \Rightarrow c-3 = 5.3 \Rightarrow \boxed{c = 8.3}$$

2. (30=10+4+8+8 points) Let  $Y$  be a continuous random variable with density function,  $f(y) = \begin{cases} \frac{1}{4}y + \frac{1}{4}, & -1 < y \leq 1 \\ -y + 2, & 1 < y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$ .

- Find the distribution function,  $F(y)$ , of  $Y$ .
- Find  $P(-1/2 < Y \leq 3/2)$ .
- Find  $P(Y \leq 3/2 | Y > -1/2)$ .
- Find the expectation of  $Y$ .



a. consider 4 cases:  $F(y) = \int_{-\infty}^y f(t) dt$

$$-\infty < y \leq -1: F(y) = \int_{-\infty}^y 0 dt = 0$$

$$-1 < y \leq 1: F(y) = F(-1) + \int_{-1}^y \left(\frac{1}{4}t + \frac{1}{4}\right) dt = 0 + \left[\frac{t^2}{8} + \frac{t}{4}\right]_{-1}^y = \frac{y^2}{8} + \frac{y}{4} + \frac{1}{8}$$

$$1 < y \leq 2: F(y) = F(1) + \int_1^y (-t + 2) dt = \frac{1}{2} + \left[-\frac{t^2}{2} + 2t\right]_1^y = -\frac{y^2}{2} + 2y - 1$$

$$2 < y < \infty: F(y) = F(2) + \int_2^y 0 dt = 1 + 0 = 1$$

$$\underline{b.} \quad P\left(-\frac{1}{2} < Y \leq \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(-\frac{1}{2}\right)$$

$$= \left[-\frac{1}{2}\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) - 1\right] - \left[\frac{1}{8}\left(-\frac{1}{2}\right)^2 + \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\right]$$

$$= \left[-\frac{9}{8} + 3 - 1\right] - \left[\frac{1}{32} - \frac{1}{8} + \frac{1}{8}\right] = \frac{27}{32} = 0.84375$$

$$\underline{c.} \quad P\left(Y \leq \frac{3}{2} | Y > -\frac{1}{2}\right) = \frac{P\left(Y \leq \frac{3}{2} \cap Y > -\frac{1}{2}\right)}{P\left(Y > -\frac{1}{2}\right)} = \frac{P\left(-\frac{1}{2} < Y \leq \frac{3}{2}\right)}{1 - P\left(Y \leq -\frac{1}{2}\right)}$$

$$= \frac{27/32}{1 - F(-1/2)} = \frac{27/32}{1 - 1/32} = \frac{27}{31} = 0.87097$$

$$\underline{d.} \quad E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \int_{-\infty}^{-1} 0 dy + \int_{-1}^1 y \left(\frac{1}{4}y + \frac{1}{4}\right) dy + \int_1^2 y(-y + 2) dy + \int_2^{\infty} 0 dy$$

$$= \int_{-1}^1 \left(\frac{1}{4}y^2 + \frac{1}{4}y\right) dy + \int_1^2 (-y^2 + 2y) dy$$

$$= \left[\frac{y^3}{12} + \frac{y^2}{8}\right]_{-1}^1 + \left[-\frac{y^3}{3} + y^2\right]_1^2$$

$$= \left(\frac{1}{12} + \frac{1}{8}\right) - \left(-\frac{1}{12} + \frac{1}{8}\right) + \left(-\frac{8}{3} + 4\right) - \left(-\frac{1}{3} + 1\right)$$

$$= \frac{2}{12} - \frac{7}{3} + 3 = \frac{1 - 14 + 18}{6} = \frac{5}{6}$$

3. (15 points) In certain experiments, the error made in determining the density of a substance is a random variable having a uniform density over the interval  $(-0.015, 0.015)$ . Find the probability that such an error will be between  $-0.002$  and  $0.003$ .

let  $Y = \text{error made}$ . Then  $Y \sim U(-0.015, 0.015)$ ,  
 so  $f(y) = \begin{cases} \frac{1}{0.030} = \frac{100}{3}, & -0.015 < y < 0.015 \\ 0, & \text{elsewhere} \end{cases}$

$P(-0.002 < Y < 0.003) = \text{shaded area}$   
 $= \frac{100}{3} \times (0.003 - (-0.002)) = \left(\frac{100}{3}\right)\left(\frac{5}{1000}\right) = \frac{1}{6}$   
 using geometry, or  
 $= \int_{-0.002}^{0.003} \frac{100}{3} dy = \frac{100}{3} y \Big|_{-0.002}^{0.003} = \frac{1}{6}$   
 using calculus.

4. (15 points) The mileage (in thousands of miles) which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with mean 40. Find the probability that one of these tires will last at most 30 thousand miles.

let  $Y = \text{lifetime (in thousands of miles) of one of these tires}$ .  
 Then  $Y \sim \text{Exponential}(40)$ , so

$$f(y) = \begin{cases} \frac{1}{40} e^{-y/40}, & 0 < y \\ 0, & \text{elsewhere.} \end{cases}$$

$$P(Y \leq 30) = \int_{-\infty}^{30} f(y) dy = \int_0^{30} \frac{1}{40} e^{-y/40} dy$$

$$\begin{array}{l} u = -y/40 \\ du = -\frac{1}{40} dy \end{array}$$

$y$	$u$
0	0
30	$-\frac{3}{4}$

$$= \int_0^{-3/4} (-e^u) du = -e^u \Big|_0^{-3/4}$$

$$= -e^{-3/4} + e^0 = 1 - e^{-3/4} = 1 - 0.4724 = 0.5276$$

5. (5 points) A random variable,  $Y$ , has density function,  $f(y) = 3e^{-3y}$ ,  $0 < y < \infty$ . What is the distribution of  $Y$ ?

What are  $E(Y)$  and  $V(Y)$ ?

$Y$  has an exponential distribution with  $\beta = \frac{1}{3}$ , or a gamma distribution with  $\alpha = 1$  and  $\beta = \frac{1}{3}$ .

$E(Y) = \beta = \frac{1}{3}$ ;  $V(Y) = \beta^2 = \frac{1}{9}$  (using the exponential distribution).

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6. (5 points) A random variable,  $Y$ , has moment-generating function,  $m(t) = e^{3t+2t^2}$ . What is the distribution of  $Y$ ?

What are  $E(Y)$  and  $V(Y)$ ?

$Y$  has a normal distribution with  $\mu = 3$  and  $\frac{\sigma^2}{2} = 2$  or  $\sigma^2 = 4$ .  $E(Y) = \mu = 3$ ;  $V(Y) = \sigma^2 = 4$ .

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7. (3 points) A random variable,  $Y$ , has a gamma distribution with parameters  $\alpha$  and  $\beta$ .  $E(Y) = 6$  and  $V(Y) = 12$ .

What are the values of  $\alpha$  and  $\beta$ ?

Since  $Y \sim \text{Gamma}(\alpha, \beta)$ ,  $6 = E(Y) = \alpha\beta$   
and  $12 = V(Y) = \alpha\beta^2$ .

Thus  $\beta = \frac{\alpha\beta^2}{\alpha\beta} = \frac{12}{6} = 2$ , and  $\alpha = \frac{6}{\beta} = \frac{6}{2} = 3$ .

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8. (5 points) Prove that  $\int_0^{\infty} y^6 e^{-2y} dy = \frac{45}{8}$ .

The integrand is the variable part of the density of a gamma distribution with  $\alpha - 1 = 6$  or  $\alpha = 7$ , and  $-2y = -y/\beta$  or  $\beta = \frac{1}{2}$ .

Thus  $1 = \int_0^{\infty} \frac{1}{\Gamma(7)(\frac{1}{2})^7} y^6 e^{-2y} dy$

$$\Rightarrow \int_0^{\infty} y^6 e^{-2y} dy = \Gamma(7)(\frac{1}{2})^7 = \frac{6!}{128} = \frac{720}{128} = \frac{45}{8}$$