Show all your work. **50 points maximum score 60 minutes** ***The test is printed on both sides of the paper***

1. **(8 points) What is the probability that an IRS auditor will catch at most two income tax returns with illegitimate deductions if she randomly selects five returns from among 15 returns, of which nine contain illegitimate deductions? First define an appropriate random variable and state its distribution.**

Let Y = # returns among the five she selects which have illegitimate deductions. Then Y has a hypergeometric distribution with $N = 15$, $r = 9$, and $n = 5$. We are asked for

$$P(Y \le 2) = P(Y=0) + P(Y=1) + P(Y=2)$$

$$= \frac{\binom{9}{0}\binom{6}{5}}{\binom{15}{5}} + \frac{\binom{9}{1}\binom{6}{4}}{\binom{15}{5}} + \frac{\binom{9}{2}\binom{6}{3}}{\binom{15}{5}} = \frac{(1)(6)+(9)(15)+(36)(20)}{\left[\frac{15\cdot 14\cdot 13\cdot 12\cdot 11}{5\cdot 4\cdot 3\cdot 2\cdot 1}\right]}$$

$$= \frac{861}{3003} = 0.2867 \approx 29\% .$$

2. (18 points) A discrete random variable, Y, has probability function, $p(y)$, given by the table below.
   a. Find $P(Y \ge 3)$.
   b. Find $P(Y \ge 3 \mid Y \text{ is odd})$.
   c. Find $E(Y)$.
   d. Find $V(Y)$.
   e. Find $E(\sqrt{Y})$.
   f. Find the moment generating function, $m(t)$ of Y.

y	0	1	4	9	16	sum
$p(y)$	0.3	0.3	0.2	0.1	0.1	1
$y\,p(y)$	0	0.3	0.8	0.9	1.6	$3.6 = E(Y)$
$y^2 p(y)$	0	0.3	3.2	8.1	25.6	$37.2 = E(Y^2)$
$\sqrt{y}\,p(y)$	0	0.3	0.4	0.3	0.4	$1.4 = E(\sqrt{Y})$
$e^{ty}p(y)$	$0.3e^{0t}$	$0.3e^{t}$	$0.2e^{4t}$	$0.1e^{9t}$	$0.1e^{16t}$	$m(t)$

(a) $P(Y \ge 3) = p(4) + p(9) + p(16) = 0.2 + 0.1 + 0.1 = 0.4$

(b) $P(Y \ge 3 \mid Y \text{ odd}) = \dfrac{P(Y \ge 3 \text{ and } Y \text{ odd})}{P(Y \text{ odd})} = \dfrac{p(9)}{p(1)+p(9)} = \dfrac{0.1}{0.3+0.1} = \dfrac{1}{4}$

(c) $E(Y) = \sum_y y\,p(y) = 3.6$ by the 3rd row of the table above.

(d) $E(Y^2) = \sum_y y^2 p(y) = 37.2$ by the 4th row of the table

so $V(Y) = E(Y^2) - [E(Y)]^2 = 37.2 - (3.6)^2 = 24.24$

(e) $E(\sqrt{Y}) = \sum_y \sqrt{y}\,p(y) = 1.4$ by the 5th row of the table

(f) $m(t) = E(e^{tY}) = \sum_y e^{ty} p(y)$

$m(t) = 0.3e^{0} + 0.3e^{t} + 0.2e^{4t} + 0.1e^{9t} + 0.1e^{16t}$

In the following three problems, a discrete random variable is described. Tell to which family of distributions the random variable belongs (binomial. geometric, negative binomial, hypergeometric, discrete uniform) and give the values of any parameters (e.g., if it is binomial, give the values of n an p). Each of these problems is worth 2 points.

3. The random variable, Y, has moment generating function $m(t) = e^{2e^t - 2}$.

   The random variable, Y, has a Poisson distribution with parameter(s) $\lambda = 2$.

4. A multiple-choice test consists of 30 questions, each of which has four choices (a, b, c, d), only one of which is correct. A person taking the test answers each question by random guessing – choosing a, b, c, or d with equal probability. Let Y denote the number of questions he correctly answers.

   The random variable, Y, has a binomial distribution with parameter(s) $n = 30$ and $p = \frac{1}{4} = 0.25$.

5. Suppose the person in problem 4 is taking a computerized version of the multiple-choice test, which requires him to keep answering questions until he gets 20 correct, at which time the test is over. He still guesses randomly. Let the random variable, X, denote the number of questions he answers.

   The random variable, X, has a negative binomial distribution with parameter(s) $r = 20$ and $p = \frac{1}{4} = 0.25$.

6. (6 points) A random variable, Y has mean -4 and variance 5. Find $E(3Y+7)$ and $V(3Y+7)$.

Given $E(Y) = -4$ and $V(Y) = 5$.

$E(3Y+7) = 3E(Y) + 7 = (3)(-4) + 7 = -5$

$V(3Y+7) = 3^2 V(Y) = (9)(5) = 45.$

7. (6 points) A random variable, Y has moment generating function $m(t) = 0.3e^{2t} + 0.5e^{4t} + 0.2e^{6t}$. ***Use the moment generating function*** to find $E(Y)$ and $V(Y)$.

$m(t) = 0.3e^{2t} + 0.5e^{4t} + 0.2e^{6t}$

$m'(t) = 0.6e^{2t} + 2.0e^{4t} + 1.2e^{6t}$

$m''(t) = 1.2e^{2t} + 8.0e^{4t} + 7.2e^{6t}$

Thus $E(Y) = m'(0) = 0.6 + 2.0 + 1.2 = 3.8$

$E(Y^2) = m''(0) = 1.2 + 8.0 + 7.2 = 16.4$

$V(Y) = E(Y^2) - [E(Y)]^2 = 16.4 - (3.8)^2 = 1.96$

8. (6 points) In the daily production of a certain kind of rope, the number of defects per foot, Y, is assumed to have a Poisson distribution with $\lambda = 2$. The profit per foot (in cents) when the rope is sold is given by X, where $X = 50 - 2Y - Y^2$. Find the expected profit per foot.

Given $Y \sim$ Poisson (2), so $E(Y) = V(Y) = 2$.

The profit per foot, $X = 50 - 2Y - Y^2$. We are asked to find $E(X) = E(50 - 2Y - Y^2)$

$= 50 - 2E(Y) - E(Y^2)$

Using the standard trick,

$V(Y) = E(Y^2) - [E(Y)]^2 \Rightarrow 2 = E(Y^2) - (2)^2$

$\Rightarrow 2 = E(Y^2) - 4 \Rightarrow E(Y^2) = 6.$

Thus $E(X) = 50 - 2E(Y) - E(Y^2)$

$= 50 - 2(2) - 6 = 40$ cents.