Poisson Processes

Suppose that we are observing the successive occurrences of some event, such as the arrival of a customer at a service counter, the arrival of an alpha particle at a Geiger counter, the arrival of a call at a telephone exchange, the occurrence of an automobile accident at a particular intersection, an insurance claim being made on a group policy or on a particular type of policy, etc. That is, we are observing a process that generates a number of occurrences (arrivals, accidents, claims, etc). Suppose further that the following conditions are satisfied (these are stated informally, not in a mathematically precise manner):

a. The numbers of occurrences in non-overlapping time intervals are independent.
b. The probability of exactly one occurrence in a sufficiently short interval of length h is approximately λh, for some $\lambda > 0$ which does not depend on the interval or on h.
c. The probability of two or more occurrences in a sufficiently short interval is essentially zero.

Let $N(t)$ denote the number of occurrences in the interval $[0, t]$. Then the collection of random variables $\{N(t) \mid t \geq 0\}$ is is called a **Poisson process with rate $\boldsymbol{\lambda}$**, and the following are true:

1. $N(t)$ has a Poisson distribution with mean λt for all $t \geq 0$. Thus, λ is the average number of occurrences in the interval $[0, 1]$ and thus in any interval of length one.
2. The waiting time until the first occurrence and the waiting times between any two consecutive occurrences have exponential distributions with mean $1/\lambda$ or rate λ.

In Sections 4.7 and 9.1 of his text, *A First Course in Probability*, 7th edition, Sheldon Ross show that (1) follows from (a), (b) and (c) above [he states them more precisely].

To show (2) follows from (1), let $N(t)$ be the number of occurrences in the interval $[0, t]$, and let W be the waiting time until the first occurrence. Then, if $F(w)$ is the cdf of W and $w > 0$,

$$\begin{aligned} F(w) &= P(W \leq w) = 1 - P(W > w) \\ &= 1 - P\big(\text{no occurences in the interval}\,[0, w]\big) \\ &= 1 - P\big(N(w) = 0\big) \\ &= 1 - \frac{(\lambda w)^0 e^{-\lambda w}}{0!} \qquad\qquad \text{since } N(w) \sim \text{Poisson}(\lambda w) \\ &= 1 - e^{-\lambda w}. \end{aligned}$$

But this is just the cdf of the exponential distribution with mean $1/\lambda$. It can be shown that properties (a) – (c) imply that the waiting time between any two consecutive occurrences has the same distribution as the waiting time until the first occurrence, so these also have exponential distributions with mean $1/\lambda$.

Now suppose that we do not know that it is a Poisson process, but we observe that the waiting times between successive occurrences have independent exponential distributions with rate λ. Then $N(t)$ has a Poisson distribution with mean λt for all $t \geq 0$.

To see this, let n be a nonnegative integer, let W_1 be the waiting time until the first occurrence and for $k > 1$, let W_k denote the waiting time between the $k-1^{\text{st}}$ occurrence and the k^{th} occurrence. Then

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n + 1) \\ &= P(W_1 + \cdots + W_n \leq t) - P(W_1 + \cdots + W_{n+1} \leq t). \end{aligned}$$

Since the W_i are independent exponential random variables with common rate λ,

$W_1 + \cdots + W_n$ has a gamma distribution with parameters n and $1/\lambda$ (using WMS notation), and

$W_1 + \cdots + W_{n+1}$ has a gamma distribution with parameters $n + 1$ and $1/\lambda$ (WMS notation).

Therefore

$$\begin{aligned} P\big(N(t) = n\big) &= \int_0^t \frac{1}{\Gamma(n)\left(\frac{1}{\lambda}\right)^n} x^{n-1} e^{-\lambda x} dx \ - \int_0^t \frac{1}{\Gamma(n+1)\left(\frac{1}{\lambda}\right)^{n+1}} x^n e^{-\lambda x} dx \\ &= \int_0^t \frac{(\lambda x)^{n-1} \lambda e^{-\lambda x}}{(n-1)!} dx \ + \int_0^t \frac{(\lambda x)^n \left(-\lambda e^{-\lambda x}\right)}{n!} dx. \end{aligned}$$

Using integration by parts, $\int (\lambda x)^n \left(-\lambda e^{-\lambda x}\right) dx = uv - \int v\, du$ where

$$u = (\lambda x)^n, \qquad dv = -\lambda e^{-\lambda x} dx, \qquad du = n\lambda\,(\lambda x)^{n-1} dx, \quad v = e^{-\lambda x},$$

so

$$\int (\lambda x)^n \left(-\lambda e^{-\lambda x}\right) dx = (\lambda x)^n e^{-\lambda x} - \int n\lambda\,(\lambda x)^{n-1} e^{-\lambda x} dx,$$

and thus

$$\begin{aligned} P\big(N(t) = n\big) &= \int_0^t \frac{(\lambda x)^{n-1} \lambda e^{-\lambda x}}{(n-1)!} dx \ + \left[\frac{(\lambda x)^n e^{-\lambda x}}{n!}\right]_0^t - \int_0^t \frac{n\lambda\,(\lambda x)^{n-1} e^{-\lambda x}}{n!} dx \\ &= \int_0^t \frac{(\lambda x)^{n-1} \lambda e^{-\lambda x}}{(n-1)!} dx \ + \left[\frac{(\lambda t)^n e^{-\lambda t} - 0}{n!}\right] - \int_0^t \frac{(\lambda x)^{n-1} \lambda e^{-\lambda x}}{(n-1)!} dx \\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \end{aligned}$$

Thus, $N(t)$ has a Poisson distribution with mean λt, as was to be shown.

Summary. Suppose that we are observing occurrences of some type of event. Let $N(t)$ denote the number of events that occur by time t. Assume that the waiting time before the first occurrence of the event and the waiting times between any two consecutive occurrences of the event are independent random variables with the same distribution, and let W denote this distribution. Then

$$N(t) \sim \text{Poisson}(\lambda t) \text{ if and only if } W \sim \text{Exp}(\lambda),$$

and, in this case, the average number of occurrences in a unit interval equals the rate at which the events occur; that is, $E[N(1)] = \lambda =$ the rate of $W = 1/E[W]$.