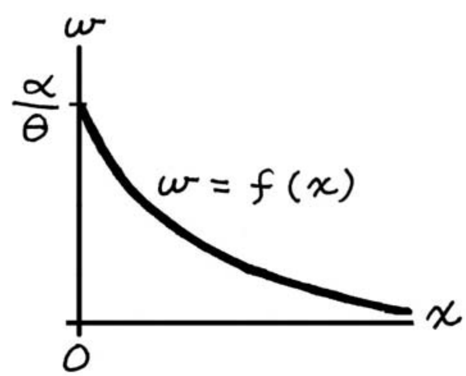
The Pareto Distributions

The Pareto distributions were initially used to model income distributions, but they are also useful in modeling many types of accident claims in casualty insurance.

Suppose that a policy has a deductible of θ. If there is a loss in excess of the deductible, let X denote the amount by which the loss exceeds the deductible, so that $X+\theta$ is the amount of the loss. Then we say that X has a Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$ if:

$$f(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \dfrac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, & \text{if } x > 0 \end{cases}$$



$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1-\left(\dfrac{\theta}{x+\theta}\right)^\alpha, & \text{if } x > 0 \end{cases}$$

and

$$S(x) = \begin{cases} 1, & \text{if } x \le 0 \\ \left(\dfrac{\theta}{x+\theta}\right)^\alpha, & \text{if } x > 0. \end{cases}$$

These functions are the pdf, the cdf, and the survival function of a continuous probability distribution since $F(x)$ is a non-decreasing, continuous function with limit 0 at $-\infty$ and with limit 1 at ∞, $S(x) = 1 - F(x)$, and $f(x) = F'(x)$ for all $x \ne 0$.

The graphs of the Pareto densities are similar to those of the exponential densities. However, one important difference between the two families of distributions is the size of the tail: for any exponential – Pareto pair with the same mean, the tail of the Pareto is heavier than the tail of the corresponding exponential. This is often what is needed to model many types of accident claims.

The Mean and Variance of $X \sim$ Pareto(α, θ) :

If $\alpha > 1$, $E(X) = \dfrac{\theta}{\alpha-1}$. (The mean is not defined for $\alpha \le 1$.)

If $\alpha > 2$, $V(X) = \dfrac{\alpha\theta^2}{(\alpha-2)\,(\alpha-1)^2}$. (The variance is not defined for $\alpha \le 2$.)

The Failure Rate or Hazard Rate for $X \sim$ Pareto(α, θ) :

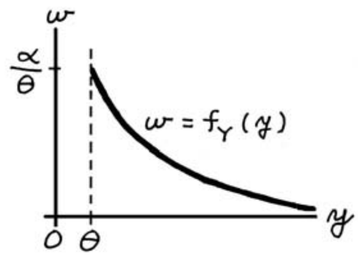
$$\lambda(t) = \frac{f(t)}{1-F(x)} = \frac{\alpha}{x+\theta}, \quad x > 0.$$

Alternate Definition of the Pareto Distributions

Some authors use the following definition for a Pareto distribution:

A random variable Y has a Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$ if its density function is

$$f_Y(y) = \begin{cases} 0, & \text{if } y \le \theta \\ \dfrac{\alpha}{\theta}\left(\dfrac{\theta}{y}\right)^{\alpha+1}, & \text{if } y > \theta. \end{cases}$$



With this definition,

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le \theta \\ 1 - \left(\dfrac{\theta}{y}\right)^{\alpha}, & \text{if } y > \theta \end{cases} \qquad \text{and} \qquad S_Y(y) = \begin{cases} 1, & \text{if } y \le \theta \\ \left(\dfrac{\theta}{y}\right)^{\alpha}, & \text{if } y > \theta. \end{cases}$$

Further, if $X = Y - \theta$, X has the Pareto distribution as it was previously defined, so that

$$E(Y) = E(X+\theta) = E(X) + \theta = \frac{\alpha\theta}{\alpha - 1}, \text{ provided } \alpha > 1,$$

and

$$V(Y) = V(X+\theta) = V(X) = \frac{\alpha\theta^2}{(\alpha-2)\,(\alpha-1)^2}, \text{ provided } \alpha > 2.$$

Interpretations of the Two Definitions when Modeling Claims on Policies with Deductibles

Assume that an insurance policy has a deductible of θ, and **assume** that there is a loss in excess of the deductible. Let

Y = the insured's loss, and

X = the insured's claim amount = loss – deductible = $Y - \theta$.

Then, if we use a Pareto distribution with parameters α and θ to model X and Y,

$$S_X(x) = P(X \ge x) = P(\text{claim} \ge x \mid \text{claim} > 0), \text{ and}$$

$$S_Y(y) = P(Y \ge y) = P(\text{loss} \ge y \mid \text{loss} > \theta).$$

Notice that both X and Y model conditional probability distributions.