Joint Moment-Generating Functions

Definition. If X and Y are jointly distributed random variables, their **joint moment-generating function** is defined to be

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right)$$

provided this expectation is defined for all (t_1, t_2) in an open disk about the origin.

A similar definition is made for the moment-generating function of a multivariate distribution $(X_1, X_2, \dots, X_n)$.

Some properties of the moment-generating function of a bivariate distribution follow.

1. $$E\left[X^n\right] = \left.\frac{\partial^n M_{X,Y}(t_1, t_2)}{\partial t_1^n}\right|_{(t_1, t_2)=(0,0)}$$

$$E\left[Y^n\right] = \left.\frac{\partial^n M_{X,Y}(t_1, t_2)}{\partial t_2^n}\right|_{(t_1, t_2)=(0,0)}$$

and, in general,

$$E\left[X^m Y^n\right] = \left.\frac{\partial^{m+n} M_{X,Y}(t_1, t_2)}{\partial t_1^m \, \partial t_2^n}\right|_{(t_1, t_2)=(0,0)}.$$

2. $$M_X(t) = M_{X,Y}(t, 0) \quad \text{and}$$

$$M_Y(t) = M_{X,Y}(0, t).$$

3. If $M_{X,Y}(t_1, t_2)$ exists, then the random variables X and Y are independent if, and only if,

$$M_{X,Y}(t_1, t_2) = M_X(t_1)\, M_Y(t_2).$$