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# HOMEWORK SOLUTIONS

## COMPUTING PROBS. & EXP. BY CONDITIONING

1. Three coins; P(heads) = 0.3, 0.5, 0.7, respectively. A coin is randomly selected and flipped twice. $N$ = # of heads on the two flips,
Let $X$ = coin selected (1, 2, or 3).
Want (a) $P(N=n)$ for $n = 0, 1, 2$
(b) $E(N)$.

In both parts, condition on $X$, the coin selected.

(a) $$P(N=n) = \sum_{i=1}^{3} P(N=n \mid X=i)\, P(X=i)$$

But conditional on $X=i$, $N$ is a binomial RV with 2 trials and probability of success 0.3 if $X=1$, 0.5 if $X=2$, and 0.7 if $X=3$.
Thus

$$\begin{aligned} P(N=n) = & \binom{2}{n}(0.3)^n(0.7)^{2-n}\cdot\left(\tfrac{1}{3}\right) && \text{coin 1} \\ & + \binom{2}{n}(0.5)^n(0.5)^{2-n}\cdot\left(\tfrac{1}{3}\right) && \text{coin 2} \\ & + \binom{2}{n}(0.7)^n(0.3)^{2-n}\cdot\left(\tfrac{1}{3}\right) && \text{coin 3} \end{aligned}$$

for $n = 0, 1, 2$.

Calculating these for each value of $N$, we obtain

$$P(N=0) = \tfrac{1}{3}\binom{2}{0}\left[(0.3)^0(0.7)^2 + (0.5)^0(0.5)^2 + (0.7)^0(0.3)^2\right] = 0.27667$$

$$P(N=1) = \tfrac{1}{3}\binom{2}{1}\left[(0.3)^1(0.7)^1 + (0.5)^1(0.5)^1 + (0.7)^1(0.3)^1\right] = 0.44667$$

$$P(N=2) = \tfrac{1}{3}\binom{2}{2}\left[(0.3)^2(0.7)^0 + (0.5)^2(0.5)^0 + (0.7)^2(0.3)^0\right] = 0.27667$$

(e) $E(N) = E[E(N|X)]$ by the double expectation theorem. The outer expectation is with respect to X, the coin selected, so

$$E[E(N|X)] = \sum_{i=1}^{3} E(N|X=i)\,P(X=i)$$

Since $P(X=i) = \frac{1}{3}$ for $i=1,2,3$, and conditional on $X=i$, N is a binomial RV with 2 trials and P(success) = 0.3, 0.5, 0.7 for X = 1, 2, 3, respectively, we get

$$\begin{aligned} E[E(N|X)] &= E(N|X=1)\,P(X=1) && \text{coin 1} \\ &\quad + E(N|X=2)\,P(X=2) && \text{coin 2} \\ &\quad + E(N|X=3)\,P(X=3) && \text{coin 3} \end{aligned}$$

$$= \underbrace{\left[2(0.3)\cdot\tfrac{1}{3}\right]}_{\text{coin 1}} + \underbrace{\left[2(0.5)\cdot\tfrac{1}{3}\right]}_{\text{coin 2}} + \underbrace{\left[2(0.7)\cdot\tfrac{1}{3}\right]}_{\text{coin 3}}$$

$$= 1$$

2. $U \sim U(0,1)$. We are told that $n$ trials are to be performed and, conditional on the value, $u$, of $U$, the trials are independent with common $P(\text{success}) = u$. Thus, conditional on $U=u$, the number, $X$, of successes on the $n$ trials is a binomial random variable, $(X|U=u) \sim bin(n,u)$

Thus

$$E[X \mid U=u] = nu$$
$$V[X \mid U=u] = nu(1-u).$$

We are asked for $E[X]$ and $V[X]$. Condition on $U=u$ and use the Double Expectation Theorem and the Conditional Variance Theorem:

$$E[X] = E[E(X|U)] = E[nU] = nE[U]$$

$$= n \cdot \left(\tfrac{1}{2}\right) \quad \text{since } U \sim U(0,1) \Rightarrow E(U) = \tfrac{1}{2}$$
$$= n/2$$

$$V[X] = V[E(X|U)] + E[V(X|U)]$$

$$= V[nU] + E[nU(1-U)]$$

$$= n^2 V[U] + nE[U - U^2]$$

$$= n^2 V[U] + nE[U] - nE[U^2]$$

Since $U \sim U(0,1)$,

$$E[U] = \frac{1}{2}$$

$$V[U] = \frac{1}{12}$$

and $E[U^2] = V[U] + (E[U])^2 = \frac{1}{12} + \left(\frac{1}{2}\right)^2 = \frac{1}{3}$

so

$$V[X] = n^2 V[U] + n\,E[U] - n\,E[U^2]$$

$$= n^2\left(\frac{1}{12}\right) + n\left(\frac{1}{2}\right) - n\left(\frac{1}{3}\right)$$

$$= \frac{1}{12} n^2 + \frac{1}{6} n.$$

3. Let $N$ = # of customers entering the store on the given day. Then $N \sim \text{Poisson}(\lambda = 10)$. Let $A_i$ be the amount spent by customer $i$, $1 \le i \le N$. Then $A_i \sim U(10, 100)$. We are asked for the mean and variance of the amount of money taken in by the store in one day.

Let $X = \sum_{i=1}^{N} A_i$. Then $X$ is the amount taken in by the store in one day. We want $E(X)$ and $V(X)$. Assuming that the $A_i$ are independent of each other and of $N$, we can compute them by conditioning on $N$:

$$E[X \mid N=n] = E\Big[\sum_{i=1}^{N} A_i \;\Big|\; N=n\Big]$$

$$= E\Big[\sum_{i=1}^{n} A_i \;\Big|\; N=n\Big]$$

$$= E\Big[\sum_{i=1}^{n} A_i\Big] \quad \text{since } A_i \text{ indep of } N$$

$$= \sum_{i=1}^{n} E(A_i) \quad \text{by property of } E$$

$$= \sum_{i=1}^{n} \frac{100}{2} \quad \text{since } A_i \sim U(0, 100)$$

$$= 50\, n$$

also, $V[X/N=n] = V\left[\sum_{i=1}^{N} A_i \mid N=n\right]$

$$= V\left[\sum_{i=1}^{n} A_i \mid N=n\right]$$

$$= V\left[\sum_{i=1}^{n} A_i\right] \quad \text{since } A_i \text{ indep of } N$$

$$= \sum_{i=1}^{n} V(A_i) \quad \text{since the } A_i \text{ are indep}$$

$$= \sum_{i=1}^{n} \frac{(100-0)^2}{12} \quad \text{since } A_i \sim U(0,100)$$

$$= \frac{10{,}000}{12}\, n.$$

thus,

$$E[X] = E\left[E[X/N]\right] = E[50N] = 50\,E[N]$$

$$= 50\lambda = 50(10) = 500 \quad \text{as } N \sim \text{Poisson}(10)$$

and

$$V(X) = V(E[X/N]) + E[V(X \mid N)]$$

$$= V(50N) + E\left[\frac{10{,}000}{12} N\right]$$

$$= 50^2\, V(N) + \left(\frac{10{,}000}{12}\right) E[N]$$

$$= 2500\lambda + \frac{10{,}000}{12}\lambda$$

as $N \sim \text{Poisson}(10)$

$$= (2500)(10) + \left(\frac{10{,}000}{12}\right)(10)$$

$$= \frac{100{,}000}{3}$$

alternatively, since $X = \sum_{i=1}^{N} A_i$ is a sum of a random number, $N$, of random variables, $A_i$, having common distribution $U(0, 100)$, we may use the formulas

$$E(X) = E(N)\,E(A_i), \quad \text{and}$$
$$V(X) = E(N)\,V(A_i) + V(N)\,[E(A_i)]^2.$$

Since $N \sim \text{Poisson}(10)$, $E(N) = V(N) = 10$ and since $A_i \sim U(0, 100)$

$$E(A_i) = \frac{100+0}{2} = 50, \quad \text{and}$$
$$V(A_i) = \left[\frac{(100-0)^2}{12}\right] = \frac{10{,}000}{12} = \frac{2500}{3}$$

so

$$\begin{aligned} E(X) &= E(N)\,E(A_i) \\ &= (10)(50) \\ &= 500 \end{aligned}$$

and

$$\begin{aligned} V(X) &= E(N)\,V(A_i) + V(N)\left[E(A_i)\right]^2 \\ &= (10)\left(\frac{2500}{3}\right) + (10)(50)^2 \\ &= \frac{25{,}000}{3} + 25{,}000 \\ &= \frac{100{,}000}{3}. \end{aligned}$$

4. $X \sim \text{Poisson}(\lambda)$ and
$\lambda \sim$ Exponential with mean 1.
We want to show for $n = 0, 1, 2, \ldots,$

$$P(X=n) = \left(\tfrac{1}{2}\right)^{n+1}.$$

Condition on the value of $\lambda$:

$$P(X=n) = \int_{-\infty}^{\infty} P(X=n \mid \lambda) f(\lambda)\, d\lambda.$$

Since $X \sim \text{Poisson}(\lambda)$, for $\lambda \geq 0$,

$$P(X=n \mid \lambda) = \frac{\lambda^n e^{-\lambda}}{n!},$$ and since

$\lambda \sim \text{Exp}(1)$, $f(\lambda) = \begin{cases} e^{-\lambda}, & \lambda > 0 \\ 0, & \lambda \leq 0. \end{cases}$

Thus,

$$P(X=n) = \int_0^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot e^{-\lambda}\, d\lambda$$

$$= \frac{1}{n!} \int_0^{\infty} \underbrace{\lambda^n e^{-2\lambda}}_{\text{variable part of a gamma density with (in WMS notation) } \alpha = n+1,\ \beta = \frac{1}{2}} d\lambda.$$

$$P(X=n) = \frac{1}{n!} \Gamma(n+1)\left(\tfrac{1}{2}\right)^{n+1} \int_0^{\infty} \frac{\lambda^n e^{-2\lambda}}{\Gamma(n+1)\left(\frac{1}{2}\right)^{n+1}}\, d\lambda$$

$$= \frac{1}{n!}(n!)\left(\tfrac{1}{2}\right)^{n+1} \cdot 1$$

$$= \left(\tfrac{1}{2}\right)^{n+1}.$$

5. $X$ = number of traffic accidents tomorrow.
Let $Y = \begin{cases} 0, & \text{if it does not rain tomorrow} \\ 1, & \text{if it does rain tomorrow.} \end{cases}$
Then we are told:

$$p_Y(y) = \begin{cases} 0.4, & y=0 \\ 0.6, & y=1 \\ 0, & \text{elsewhere} \end{cases}$$

$$(X \mid Y=0) \sim \text{Poisson}(3)$$

$$(X \mid Y=1) \sim \text{Poisson}(9),$$

In all three parts, condition on the value of $Y$ (or on rain / no rain).

(a) $$E(X) = E\left[E[X \mid Y]\right]$$
$$= E[X \mid Y=0]\,p_Y(0) + E[X \mid Y=1]\,p_Y(1)$$
$$= (3)(0.4) + (9)(0.6)$$
$$= 6.6$$

(b) $$P(X=0) = P(X=0 \mid Y=0)\,p_Y(0) + P(X=0 \mid Y=1)\,p_Y(1)$$
$$= \left(\frac{3^0 e^{-3}}{0!}\right)(0.4) + \left(\frac{9^0 e^{-9}}{0!}\right)(0.6)$$
$$= 0.4e^{-3} + 0.6e^{-9}$$
$$= 0.01999$$

where the second equality follows since
$(X \mid Y=0) \sim \text{Poisson}(3) \Rightarrow P(X=x \mid Y=0) = \dfrac{3^x e^{-3}}{x!}$
and
$(X \mid Y=1) \sim \text{Poisson}(9) \Rightarrow P(X=x \mid Y=1) = \dfrac{9^x e^{-9}}{x!}$

5. (c) To compute $V(X)$, we must compute $E(X)$ and $E(X^2)$ by conditioning on $Y$ and then use the computing formula for the variance.

From (a), $E(X) = 6.6$.

$$E(X^2) = E[E(X^2 \mid Y)]$$

$$= E(X^2 \mid Y=0)p_Y(0) + E(X^2 \mid Y=1)p_Y(1).$$

Since $(X \mid Y=0) \sim \text{Poisson}(\lambda=3)$,

$$\begin{aligned} E(X^2 \mid Y=0) &= V(X \mid Y=0) + [E(X \mid Y=0)]^2 \\ &= \lambda + \lambda^2 \\ &= 3 + 3^2 \\ &= 12. \end{aligned}$$

Similarly, $(X \mid Y=1) \sim \text{Poisson}(\lambda=9) \Rightarrow$

$E(X^2 \mid Y=1) = 9 + 9^2 = 90.$

Thus
$$\begin{aligned} E(X^2) &= E(X^2 \mid Y=0)p_Y(0) + E(X^2 \mid Y=1)p_Y(1) \\ &= (12)(0.4) + (90)(0.6) \\ &= 58.8 \end{aligned}$$

Thus,
$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= 58.8 - (6.6)^2 \\ &= 15.24 \end{aligned}$$

6. Let $X$ = # of storms this season.
We are told that:

$$(X \mid \Lambda = \lambda) \sim \text{Poisson}(\lambda)$$
$$\Lambda \sim U(0,5).$$

Thus,
$$P(X=x \mid \Lambda=\lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x=0,1,2,\dots \\ 0, & \text{elsewhere} \end{cases}$$
and
$$f_\Lambda(\lambda) = \begin{cases} \frac{1}{5}, & 0<\lambda<5 \\ 0, & \text{elsewhere.} \end{cases}$$

We are asked for $P(X \geq 3) = 1 - P(X<3)$
$= 1 - P(X \leq 2)$.

Condition on the value of $\Lambda$:

$$P(X\geq 3) = 1 - P(X \leq 2)$$
$$= 1 - \int_{-\infty}^{\infty} P(X \leq 2 \mid \Lambda = \lambda) f_\Lambda(\lambda)\, d\lambda$$
$$= 1 - \int_0^5 \left[\frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!}\right] \cdot \frac{1}{5}\, d\lambda$$
$$= 1 - \int_0^5 \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{1}{2}\lambda^2 e^{-\lambda}\right) \cdot \frac{1}{5}\, d\lambda$$
$$= 1 - \frac{1}{5}\int_0^5 e^{-\lambda}\, d\lambda - \frac{1}{5}\int_0^5 \lambda e^{-\lambda}\, d\lambda - \frac{1}{10}\int_0^5 \lambda^2 e^{-\lambda}\, d\lambda.$$

The latter two integrals can be calulated using integration by parts.

$$\int e^{-\lambda}\,d\lambda = -e^{-\lambda} + C, \quad \text{so}$$

$$\int_0^5 e^{-\lambda}\,d\lambda = \Big[-e^{-\lambda}\Big]_{\lambda=0}^{\lambda=5} = 1 - e^{-5}$$

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$$\int \lambda e^{-\lambda}\,d\lambda = uv - \int v\,du \qquad \boxed{\begin{array}{l} u=\lambda,\ dv = e^{-\lambda}d\lambda \\ du = d\lambda,\ v = -e^{-\lambda}\end{array}}$$

$$= -\lambda e^{-\lambda} - \int(-e^{-\lambda})\,d\lambda$$

$$= -\lambda e^{-\lambda} + \int e^{-\lambda}\,d\lambda$$

$$= -\lambda e^{-\lambda} - e^{-\lambda} + C, \quad \text{so}$$

$$\int_0^5 \lambda e^{-\lambda}\,d\lambda = \Big[-\lambda e^{-\lambda} - e^{-\lambda}\Big]_{\lambda=0}^{\lambda=5}$$

$$= 1 - 5e^{-5} - e^{-5} = 1 - 6e^{-5}$$

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$$\int \lambda^2 e^{-\lambda}\,d\lambda = uv - \int v\,du \qquad \boxed{\begin{array}{l} u=\lambda^2,\ dv = e^{-\lambda}d\lambda \\ du = 2\lambda\,d\lambda,\ v = -e^{-\lambda}\end{array}}$$

$$= -\lambda^2 e^{-\lambda} - \int 2\lambda(-e^{-\lambda})\,d\lambda$$

$$= -\lambda^2 e^{-\lambda} + 2\int \lambda e^{-\lambda}\,d\lambda$$

$$= -\lambda^2 e^{-\lambda} + 2\Big[-\lambda e^{-\lambda} - e^{-\lambda}\Big] + C$$

$$= -\lambda^2 e^{-\lambda} - 2\lambda e^{-\lambda} - 2e^{-\lambda} + C, \quad \text{so}$$

$$\int_0^5 \lambda^2 e^{-\lambda}\,d\lambda = \Big[-\lambda^2 e^{-\lambda} - 2\lambda e^{-\lambda} - 2e^{-\lambda}\Big]_{\lambda=0}^{\lambda=5}$$

$$= 2 - 25e^{-5} - 10e^{-5} - 2e^{-5}$$

$$= 2 - 37e^{-5}$$

Thus,

$$P(X \geq 3) = 1 - \frac{1}{5}\int_0^5 e^{-\lambda} d\lambda - \frac{1}{5}\int_0^5 \lambda e^{-\lambda} d\lambda - \frac{1}{10}\int_0^5 \lambda^2 e^{-\lambda} d\lambda$$

$$= 1 - \frac{1}{5}(1 - e^{-5}) - \frac{1}{5}(1 - 6e^{-5}) - \frac{1}{10}(2 - 37e^{-5})$$

$$= 1 - \frac{1}{10}(6 - 51)e^{-5}$$

$$= 0.434$$