1. Let E and F be independent events with $P(E) = P(F)$ and $P(E \cup F) = \frac{1}{2}$. What is $P(E)$?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. $1 - \frac{1}{\sqrt{2}}$ D. $\frac{1}{\sqrt{2}}$

E. cannot be determined from the given information

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Let $x = P(E) = P(F)$. Then since E and F are independent, $P(E \cap F) = P(E)\,P(F) = x^2$. Thus

$$\frac{1}{2} = x + x - x^2 \Rightarrow x^2 - 2x + \frac{1}{2} = 0$$

or $2x^2 - 4x + 1 = 0$. Using the quadratic formula,

$$x = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2(2)} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4}$$

so $x = 1 \pm \frac{1}{2}\sqrt{2}$. Since $0 < x < 1$, $x = 1 - \frac{\sqrt{2}}{2}$.

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2. Urn I contains 7 red and 3 black balls, and urn II contains 4 red and 5 black balls. After a randomly selected ball is transferred from urn I to urn II, 2 balls are randomly drawn from urn II without replacement. What is the probability that both balls drawn from urn II are red?

A. $\frac{9}{20}$ B. $\frac{8}{45}$ C. $\frac{47}{100}$ D. $\frac{28}{125}$ E. $\frac{44}{225}$

$P(RR) = P(R \text{ transferred}) \cdot P(RR \mid R \text{ transferred}) + P(B \text{ transferred}) \cdot P(RR \mid B \text{ transferred})$

$$= \frac{7}{10} \cdot \frac{\binom{5}{2}}{\binom{10}{2}} + \frac{3}{10} \cdot \frac{\binom{4}{2}}{\binom{10}{2}}$$

$$= \frac{7}{10} \cdot \frac{10}{45} + \frac{3}{10} \cdot \frac{6}{45}$$

$$= \frac{88}{450} = \frac{44}{225}$$

after a ball is transferred, urn II will have
5R & 5B, if R transf.
or
4R & 6B, if B transf.

3. X is a continuous loss random variable whose 75th percentile is $\frac{1000}{3}$, and whose density function is $f(x)=\begin{cases} 0.001\lambda e^{-0.001\lambda x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$. Find λ.

A. $\ln\frac{64}{27}$ B. $\frac{4}{3}\ln\frac{3}{2}$ C. $\ln 12$ D. $\ln 64$ E. $\ln 81$

Since X has an exponential distribution with mean $\frac{1}{0.001\lambda}$, its cdf is $F(x)=\begin{cases} 0, & x\le 0 \\ 1-e^{-0.001\lambda x}, & x>0. \end{cases}$

Since its 75% percentile is $\frac{1000}{3}$, we know

$$F\left(\frac{1000}{3}\right)=\frac{3}{4} \Rightarrow 1-e^{-0.001\lambda\cdot 1000/3}=\frac{3}{4}$$

$$\Rightarrow e^{-\lambda/3}=\frac{1}{4} \Rightarrow -\lambda/3=\ln\frac{1}{4}=-\ln 4$$

$$\Rightarrow \lambda=3\ln 4=\ln 4^3=\ln 64.$$

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4. A random variable X has density function $f(x)=\begin{cases} 2(1-x), & 0\le x\le 1 \\ 0, & \text{elsewhere.} \end{cases}$

What is the probability that the larger of 2 independent observations on X will exceed $\frac{1}{2}$?

A. $\frac{4}{16}$ B. $\frac{7}{16}$ C. $\frac{8}{16}$ D. $\frac{9}{16}$ E. $\frac{12}{16}$

Let X_1 and X_2 be 2 independent observations on X, $Y=\max(X_1,X_2)$. Then we want

$$P(Y>\tfrac{1}{2})=1-P(Y\le\tfrac{1}{2})=1-P(X_1\le\tfrac{1}{2},X_2\le\tfrac{1}{2})$$
$$=1-P(X_1\le\tfrac{1}{2})P(X_2\le\tfrac{1}{2}) \text{ by independence,}$$
$$=1-[P(X\le\tfrac{1}{2})]^2.$$

But $P(X\le\frac{1}{2})=\int_0^{1/2}(2-2x)\,dx=2x-x^2\Big|_0^{1/2}$

$$=\left(1-\tfrac{1}{4}\right)-(0-0)=\tfrac{3}{4}.$$

Thus $P(Y>\frac{1}{2})=1-\left(\frac{3}{4}\right)^2=\frac{7}{16}.$

5. A loss random variable X has distribution function $F(x) = \begin{cases} 0, & x < 0 \\ 0.2 + 0.3x, & 0 \le x \le 2 \\ 1, & x > 2. \end{cases}$

An insurer will provide proportional insurance on this loss, covering fraction α of the loss ($0 < \alpha < 1$). The expected claim on the insurer is 0.5. Find α.

A. 0.25 B. 0.3 C. 0.45 D. 0.5 E. 0.65

We are told $0.5 = E(\alpha X) = \alpha E(X)$ and are asked to find $\alpha = \frac{0.5}{E(X)}$. We can calculate $E(X)$ in one of two ways, using $F(x)$ or $f(x)$. Since X has a mixed distribution [$P(X=0)=0.2$ and $P(X=2)=0.2$], it is easier to use $F(X)$. [Actually, since we are told $F(x)$ and not $f(x)$, it is easier to use $F(x)$.] So

$$E(X) = \int_0^\infty [1-F(x)]\,dx = \int_0^2 [1-(0.2+0.3x)]\,dx$$

$$= \int_0^2 (0.8-0.3x)\,dx = 0.8x - 0.15x^2 \Big|_0^2$$

$$= (1.6-0.6)-(0-0) = 1$$

so that $\alpha = \frac{0.5}{1} = 0.5$.

Using $f(x)$, we first find $f(x)$:

$$f(x) = \begin{cases} 0.2, & x=0 \text{ (discrete here)} \\ 0.3, & 0<x<2 \text{ (cont here)} \\ 0.2, & x=2 \text{ (discrete here} \\ 0, & \text{elsewhere} \end{cases}$$

So $E(X) = 0 \cdot P(X=0) + \int_0^2 x \cdot 0.3\,dx + 2 \cdot P(X=2)$

$$= 0 + 0.15x^2 \Big|_0^2 + 2(0.2)$$

$$= 0 + (0.6-0) + 0.4$$

$$= 1.$$

6. In an initial screening of job applicants, a recruiter accepts, on average, one-third of all applications for further consideration. In reviewing a collection of job applications (independent of one another), find the probability that the first application acceptable for further consideration is one of the first three applications reviewed.

A. $\frac{17}{27}$ B. $\frac{19}{27}$ C. $\frac{7}{9}$ D. $\frac{23}{27}$ E. $\frac{25}{27}$

Method 1: Let X = the number of applications reviewed before an acceptable one is found; Then $X \sim \text{Geom}(\frac{1}{3})$ and we want $P(X \leq 2)$ using the study guide's notation.

$$P(X \leq 2) = 1 - P(X \geq 3) = 1 - q^3 = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$$

Method 2: Let Y = the number of applications reviewed until an acceptable one is found (Y counts the acceptable one; X above did not). Then $Y \sim \text{Geom}(\frac{1}{3})$ and we want $P(Y \leq 3)$ using the notation of MUS.

$$P(Y \leq 3) = 1 - P(Y > 3) = 1 - q^3 = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$$

Method 3: Let B = the number of acceptable applications among the first 3 reviewed. Then $B \sim \text{Binomial}(3, \frac{1}{3})$ and we want

$$P(B \geq 1) = 1 - P(B=0) = 1 - \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = \frac{19}{27}$$

Method 4: The problem asks for the probability that at least one of the first 3 applications reviewed is acceptable

$= 1 - P(\text{none of the first 3 is acceptable})$

$= 1 - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$ by independence

$= \frac{19}{27}$.

7. An urn contains 2 white marbles and 8 red marbles. A marble is drawn from the urn 100 times in succession with replacement. Using the integer correction, find the approximate probability of drawing more than 75 red marbles.

A. 0.11 B. 0.62 C. 0.75 D. 0.87 E. 0.95

Let X = # of red marbles drawn. Then $X \sim bin(100, \frac{8}{10})$ since the marbles are drawn with replacement.

Thus $E(X) = 100(\frac{8}{10}) = 80$ and $V(X) = 100(\frac{8}{10})(\frac{2}{10}) = 16$.

Thus

$$P(X>75) = P(X \geq 76) = \underbrace{P(X \geq 75.5)}_{\text{integer correction}}$$

$$= P\left(Z \geq \frac{75.5-80}{\sqrt{16}} = -1.125\right)$$

$$= P(Z \leq 1.125) = 0.8697$$

8. An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

	Type of driver	Percentage of all drivers	Probability of at least one collision
T	Teen	8%	0.15
Y	Young adult	16%	0.08
M	Midlife	45%	0.04
S	Senior	31%	0.05
	Total	100%	

Given that a driver has been involved in at least one collision (C) in the past year, what is the probability that the driver is a young adult driver?

A. 0.06 B. 0.16 C. 0.19 D. 0.22 E. 0.25

This is a Bayes' Formula Problem:

$$P(Y|C) = \frac{P(Y)P(C|Y)}{P(T)P(C|T) + P(Y)P(C|Y) + P(M)P(C|M) + P(S)P(C|S)}$$

$$= \frac{(0.16)(0.08)}{(0.08)(0.15)+(0.16)(0.08)+(0.45)(0.04)+(0.31)(0.05)}$$

$$= \frac{0.0128}{0.0120 + 0.0128 + 0.0180 + 0.0155}$$

$$= \frac{0.0128}{0.0583} = 0.22$$

9. The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, is uniformly distributed on the interval (0, 5). Let $Y = h(X)$ be the age of the machine at the time of replacement. Determine the expected age of the machine at replacement, $E(Y) = E(h(X)$.

A. 1.6 B. 2.4 C. 2.5 D. 3.2 E. 4.0

Given $X \sim U(0,5)$ and $Y = h(X) = \begin{cases} X, & X \le 4 \\ 4, & X > 4 \end{cases}$

$$E(Y) = E(h(X)) = \int_0^5 h(x) \cdot \frac{1}{5}\,dx$$

$$= \int_0^4 \frac{x}{5}\,dx + \int_4^5 \frac{4}{5}\,dx = \frac{x^2}{10}\Big|_0^4 + \frac{4x}{5}\Big|_4^5$$

$$= \left(\frac{16}{10} - 0\right) + \left(4 - \frac{16}{5}\right) = 1.6 + 4 - 3.2 = 2.4$$

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10. The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

A. $\frac{1}{3}$ B. $\frac{2}{5}$ C. $\frac{1}{2}$ D. $\frac{3}{5}$ E. $\frac{5}{6}$

$$P(N \ge 1 \mid N \le 4) = \frac{P(N \ge 1 \;\&\; N \le 4)}{P(N \le 4)}$$

$$= \frac{P(1 \le N \le 4)}{P(N \le 4)}$$

$$= \frac{(\frac{1}{2})(\frac{1}{3}) + (\frac{1}{3})(\frac{1}{4}) + (\frac{1}{4})(\frac{1}{5}) + (\frac{1}{5})(\frac{1}{6})}{(\frac{1}{1})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{3}) + (\frac{1}{3})(\frac{1}{4}) + (\frac{1}{4})(\frac{1}{5}) + (\frac{1}{5})(\frac{1}{6})}$$

$$= \frac{10 + 5 + 3 + 2}{30 + 10 + 5 + 3 + 2} = \frac{20}{50} = \frac{2}{5}$$