NAME SOLUTIONS

A380/M480 Quiz 4a February 22, 2005 20 pts. max Show all work; no work $\Rightarrow$ no credit.

1. The number of power surges in an electrical grid has a Poisson distribution with a mean of 1 power surge every 12 hours. What is the probability that there will be no more than 1 power surge in a 24-hour period?

A. $2e^{-2}$ B. $3e^{-2}$ C. $e^{-1/2}$ D. $\frac{3}{2}e^{-1/2}$ E. $3e^{-1}$

Method 1: Let X = number of power surges in a 24-hour period. Then X has a Poisson distribution with mean $2 = 2\cdot 1$ (since $24 = 2\cdot 12$). So

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$= \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} = 3e^{-2}$$

Method 2: Power surges in non-overlapping 12-hour periods are independent occurrences by the Poisson process assumptions. So if X_1 = # of surges in the first half of the 24-hour period and X_2 = the # in the second half, X_1 and X_2 are independent Poisson($\lambda = 1$) random variables, and we want $P(X_1 + X_2 = 1)$. But $X_1 + X_2 = 1$ means exactly one of the disjoint (mut. exclusive) events

$(X_1=0, X_2=0)$, $p = \frac{1^0 e^{-1}}{0!} \cdot \frac{1^0 e^{-1}}{0!} = e^{-2}$

$(X_1=0, X_2=1)$, $p = \frac{1^0 e^{-1}}{0!} \cdot \frac{1^1 e^{-1}}{1!} = e^{-2}$

or $(X_1=1, X_2=0)$, $p = \frac{1^1 e^{-1}}{1!} \cdot \frac{1^0 e^{-1}}{0!} = e^{-2}$

occurs, so

$$P(X_1 + X_2 = 1) = e^{-2} + e^{-2} + e^{-2} = 3e^{-2}.$$

2. As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represented the number of people tested until the first person with high blood pressure is found. The expected value of X is 12.5. Calculate the probability that the sixth person tested is the first one with high blood pressure.

A. 0.000 B. 0.053 C. 0.080 D, 0.316 E. 0.394

$X \sim$ Geometric (p) where $p =$ prob. a randomly selected prospective policyholder has high blood pressure.

From the wording, X is the number of the trial on which the first person with high blood pressure is found.

Thus $E(X) = \dfrac{1}{p} = 12.5 = \dfrac{25}{2} \Rightarrow p = \dfrac{2}{25} = 0.08.$

We are asked to calculate

$$P(X=6) = (1-p)^5 \cdot p = (0.92)^5 \cdot (0.08)$$

$$= 0.0527$$

3. Let X be a random variable with moment-generation function

$$M(t) = \left(\frac{2+e^t}{3}\right)^9.$$

Calculate the variance of X.

A. 2 B. 3 C. 8 D. 9 E. 11

Method 1: $M(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$ is the mgf of a binomial distribution with $n=9$ and $p=\frac{1}{3}$, so $V(X) = np(1-p) = 9\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2$.

Method 2:

$$M'(t) = 9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}$$

$$M''(t) = 72\left(\frac{2+e^t}{3}\right)^7 \cdot \left(\frac{e^t}{3}\right)^2 + 9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}$$

so $E(X) = M'(0) = 9\left(\frac{2+1}{3}\right)^8 \cdot \frac{1}{3} = 3$

$$E(X^2) = M''(0) = 72\left(\frac{2+1}{3}\right)^7 \cdot \left(\frac{1}{3}\right)^2 + 9\left(\frac{2+1}{3}\right)^8 \cdot \frac{1}{3}$$

$$= 72 \cdot 1 \cdot \frac{1}{9} + 9 \cdot 1 \cdot \frac{1}{3}$$

$$= 8 + 3 = 11$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= 11 - 3^2 = 11 - 9 = 2$$

4. A random sample of size 6 is selected WITH replacement from an urn that contains 10 red, 5 white, and 5 blue balls. What is the probability that the sample contains 2 balls of each color?

A. $\frac{1}{1024}$ B. $\frac{1}{646}$ C. $\frac{45}{512}$ D. $\frac{75}{646}$ E. $\frac{45}{64}$

Method 1. Let X_1 = # red in sample, X_2 = # white in sample, and X_3 = # blue in sample. Then (X_1, X_2, X_3) has a multinomial distribution (since sampling with replacement) with parameters $n=6$, $p_1 = \frac{10}{20} = \frac{1}{2}$, $p_2 = \frac{5}{20} = \frac{1}{4}$, and $p_3 = \frac{5}{20} = \frac{1}{4}$. Thus,

$$P(X_1 = 2, X_2 = 2, X_3 = 2) = \binom{6}{2\ 2\ 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^2$$

$$= \frac{6!}{2!2!2!} \cdot \frac{1}{4} \cdot \frac{1}{16} \cdot \frac{1}{16} = \frac{\overset{3}{\cancel{6}} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}}{2 \cdot \cancel{4} \cdot \cancel{2}} \cdot \frac{1}{\cancel{4}} \cdot \frac{1}{16} \cdot \frac{1}{16}$$

$$= \frac{45}{512}$$

Method 2. Think "fill blanks": $\underset{\text{ball \#1}}{\_\_}\ \underset{\#2}{\_\_}\ \underset{\#3}{\_\_}\ \underset{\#4}{\_\_}\ \underset{\#5}{\_\_}\ \underset{\#6}{\_\_}$

The number of ways to fill these 6 blanks with 2 balls of each color is $\binom{6}{2\ 2\ 2} = \binom{6}{2}\binom{4}{2}\binom{2}{2}$.

Any sequence of 2R, 2W, 2B has the same probability of occurring, namely $\left(\frac{10}{20}\right)^2\left(\frac{5}{20}\right)^2\left(\frac{5}{20}\right)^2$.

Thus the probability is

$$\binom{6}{2\ 2\ 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^2$$

as before.